

Application of the lace expansion to the φ^4 model

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Abstract

Using the Griffiths-Simon construction of the φ^4 model and the lace expansion for the Ising model, we prove that, if the strength $\lambda \geq 0$ of nonlinearity is sufficiently small for a large class of short-range models in dimensions $d > 4$, then the critical φ^4 two-point function $\langle \varphi_o \varphi_x \rangle_{\mu_c}$ is asymptotically $|x|^{2-d}$ times a model-dependent constant, and the critical point is estimated as $\mu_c = \hat{\mathcal{J}} - \frac{\lambda}{2} \langle \varphi_o^2 \rangle_{\mu_c} + O(\lambda^2)$, where $\hat{\mathcal{J}}$ is the massless point for the Gaussian model.

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1 Introduction and the main results

The (lattice) φ^4 model is a pedagogical yet nontrivial model in scalar field theory. It is also considered to be an interface model defined by a Hamiltonian having a quartic self-energy term. (See, e.g., [6] for recent development in another class of interface models, called gradient fields.) If that quartic term is absent, then it becomes the Gaussian model

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and its two-point function satisfies the same convolution equation as the random-walk's Green function. In particular, for the massless Gaussian model (or often called Gaussian free fields), the two-point function decays as a multiple of $|x|^{2-d}$ as $|x| \rightarrow \infty$ when $d > 2$.

On the other hand, the φ^4 two-point function is known to satisfy a nonlinear equation, called the Schwinger-Dyson equation. The nonlinearity is due to involvement of four-spin expectations. This implies that, in order to find the exact expression for the two-point function, we must also know the exact expressions for four-spin expectations. In general, the Schwinger-Dyson equation for $2n$ -spin expectations involves $(2n+2)$ -spin expectations. Therefore, it is seemingly impossible to solve those infinitely many simultaneous equations to find the exact expression for the two-point function.

Instead of solving those simultaneous equations, there have been many useful ideas to study the phase transition and critical behavior for the φ^4 model. Among those are the use of reflection positivity [9, 10] and correlation inequalities obtained by the random-current representation [1] and the random-walk representation [2, 3]. They imply that, for the nearest-neighbor model in dimensions $d > 2$, there is a nontrivial critical point $\mu_c \in \mathbb{R}$ such that the two-point function $\langle \varphi_o \varphi_x \rangle_\mu$ is bounded above by a multiple of $|x|^{2-d}$ uniformly in $\mu > \mu_c$, and therefore all critical exponents take on their mean-field values for $d > 4$ [1, 8, 23] (see also [7] and references therein). Moreover, for the nearest-neighbor model, the rigorous renormalization group approach based on the block-spin transformation [11] may identify an asymptotic expression for the critical two-point function $\langle \varphi_o \varphi_x \rangle_{\mu_c}$, which is presumably $C|x|^{2-d}$ as $|x| \rightarrow \infty$ for some constant $C \in (0, \infty)$. However, as far as we are aware, there have been no such strong results for the φ^4 two-point function.

For the Ising model, which is considered to be in the same universality class as the φ^4 model, we have been able to show [20] that, not only for the nearest-neighbor model but also for a large class of spread-out models which do not necessarily satisfy reflection positivity, the critical Ising two-point function is asymptotically a model-dependent multiple of $|x|^{2-d}$, if the dimension d or the range of spin-spin coupling is sufficiently large. The proof is based on the lace expansion, which was first applied to weakly self-avoiding walk [4] and then developed for lattice trees and lattice animals [14], percolation [15], oriented percolation [18] and the contact process [19]. The asymptotic behavior of the critical two-point function for each model is proved in [12, 13, 20]. The methodology has been extended to cover the case of power-law decaying spin-spin coupling [5] (see also [16] for results in the Fourier space).

In this paper, we apply the lace expansion for the Ising model to prove asymptotic behavior of the φ^4 two-point function. In order to do so, we first use the Griffiths-Simon construction [21] to approximate each φ^4 spin by a sum of N Ising-spin variables. Then, we investigate the lace-expansion coefficients and determine the right scaling in powers of N . As a result, we prove the expected asymptotic behavior of the critical two-point function $\langle \varphi_o \varphi_x \rangle_{\mu_c}$ for a large class of short-range models on \mathbb{Z}^d with $d > 4$, if the strength $\lambda \geq 0$ of nonlinearity is sufficiently small. As a byproduct, we obtain the λ -expansion of the critical point μ_c up to $O(\lambda^2)$ around the massless point for the Gaussian model.

Before showing the precise statement of the above result, we first provide the precise definition of the model.

1.1 The φ^4 model

For a finite set $\Lambda \subset \mathbb{Z}^d$, we define the Hamiltonian \mathcal{H}_Λ on the space \mathbb{R}^Λ of spin configurations as follows: for $\varphi \equiv (\varphi_x)_{x \in \Lambda} \in \mathbb{R}^\Lambda$,

$$\mathcal{H}_\Lambda(\varphi) = -\frac{1}{2} \sum_{u \neq v \in \Lambda} \mathcal{J}_{u,v} \varphi_u \varphi_v + \sum_{v \in \Lambda} \left(\frac{\mu}{2} \varphi_v^2 + \frac{\lambda}{4!} \varphi_v^4 \right), \quad (1.1)$$

where $\mathcal{J}_{u,v}$ is a nonnegative, translation-invariant and \mathbb{Z}^d -symmetric coupling function: $\mathcal{J}_{u,v} = \mathcal{J}_{o,v-u} \equiv \mathcal{J}(v-u) \geq 0$. We also assume $\mathcal{J}(o) = 0$ and that \mathcal{J} is summable:

$$\hat{\mathcal{J}} \equiv \sum_{v \in \mathbb{Z}^d} \mathcal{J}(v) < \infty. \quad (1.2)$$

The parameter $\mu \in \mathbb{R}$ plays the role of temperature, while $\lambda \geq 0$ is the intensity of nonlinearity. We call the model Gaussian if $\lambda = 0$, and in that case, we can rewrite the Hamiltonian as

$$\mathcal{H}_\Lambda^{\lambda=0}(\varphi) = \frac{\hat{\mathcal{J}}}{2}(\varphi, -\Delta\varphi) + \frac{\mu - \hat{\mathcal{J}}}{2} |\varphi|^2, \quad (1.3)$$

where (\cdot, \cdot) is the inner product and Δ is the lattice Laplacian defined by the transition probability $\mathcal{J}(x)/\hat{\mathcal{J}}$. The first term on the right-hand side represents the kinetic energy, while the second term represents the potential. We call the Gaussian model massless if the potential is zero (i.e., $\mu = \hat{\mathcal{J}}$).

The key quantity we are interested in is the two-point function $\langle \varphi_o \varphi_x \rangle_\mu$, which is the increasing limit as $\Lambda \uparrow \mathbb{Z}^d$ (due to the second Griffiths inequality, e.g., [7]) of the finite-volume expectation $\langle \varphi_o \varphi_x \rangle_{\mu, \Lambda}$:

$$\langle \varphi_o \varphi_x \rangle_\mu = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \varphi_o \varphi_x \rangle_{\mu, \Lambda} \equiv \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\int_{\mathbb{R}^\Lambda} \varphi_o \varphi_x e^{-\mathcal{H}_\Lambda(\varphi)} d\varphi}{\int_{\mathbb{R}^\Lambda} e^{-\mathcal{H}_\Lambda(\varphi)} d\varphi}. \quad (1.4)$$

Due to Lebowitz' inequality [17], there exists a critical point $\mu_c \equiv \mu_c(d, \mathcal{J}, \lambda) \leq \hat{\mathcal{J}}$ such that the susceptibility

$$\chi_\mu \equiv \sum_{x \in \mathbb{Z}^d} \langle \varphi_o \varphi_x \rangle_\mu \quad (1.5)$$

is finite if and only if $\mu > \mu_c$ and diverges as $\mu \downarrow \mu_c$ (cf., Figure 1).

One of the possible approaches to investigate the two-point function is to use the result of integration by parts: $\langle \frac{\partial \mathcal{H}_\Lambda}{\partial \varphi_o} \varphi_x \rangle_\Lambda = \delta_{o,x}$. Plugging (1.1) into this identity and taking the infinite-volume limit, we obtain the Schwinger-Dyson equation

$$-\sum_v \mathcal{J}_{o,v} \langle \varphi_v \varphi_x \rangle_\mu + \mu \langle \varphi_o \varphi_x \rangle_\mu + \frac{\lambda}{3!} \langle \varphi_o^3 \varphi_x \rangle_\mu = \delta_{o,x}. \quad (1.6)$$

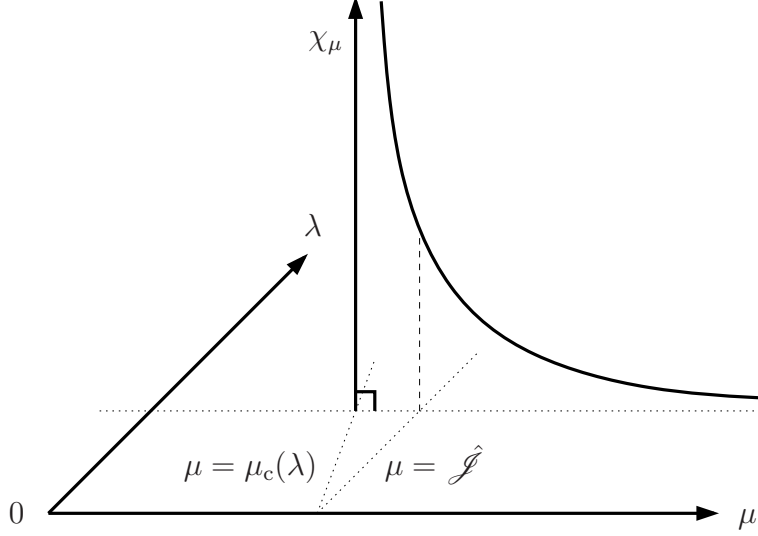


Figure 1: Divergence of the susceptibility χ_μ as $\mu \downarrow \mu_c(\lambda)$ for a fixed $\lambda > 0$ (along the horizontal dotted line). The susceptibility is still finite below the massless line $\mu = \hat{\mathcal{J}}$, whereas the potential $\mathcal{H}_\Lambda(\varphi) - \frac{\hat{\mathcal{J}}}{2}(\varphi, -\Delta\varphi)$ is not convex any more.

This immediately implies that the two-point function for the Gaussian model satisfies the same convolution equation as the random-walk's Green function generated by the 1-step distribution $\mathcal{J} / \hat{\mathcal{J}}$ with killing rate $1 - \hat{\mathcal{J}}/\mu$. Therefore, at the massless point $\mu = \hat{\mathcal{J}}$, the two-point function exhibits the same asymptotic behavior as the Green function for the underlying random walk. In this paper, in addition to the properties stated below (1.1), we assume that all moments of \mathcal{J} are finite. In particular, we define the variance

$$V = \sum_{x \in \mathbb{Z}^d} |x|^2 \frac{\mathcal{J}(x)}{\hat{\mathcal{J}}}. \quad (1.7)$$

Then, by the Cramér-Edgeworth expansion (e.g., [5, Theorem A.1]), we can estimate the n -fold convolution of $\mathcal{J} / \hat{\mathcal{J}}$ and its “derivative” as (cf., [5, (1.21) and (1.24)])

$$\frac{\mathcal{J}^{*n}(x)}{\hat{\mathcal{J}}^n} \leq \frac{O(n)}{(|x| \vee 1)^{d+2}}, \quad (1.8)$$

$$\left| \frac{\mathcal{J}^{*n}(x)}{\hat{\mathcal{J}}^n} - \frac{\mathcal{J}^{*n}(x+y) + \mathcal{J}^{*n}(x-y)}{2\hat{\mathcal{J}}^n} \right| \leq \frac{O(n)|y|^2}{(|x| \vee 1)^{d+4}} \quad [|y| \leq \tfrac{1}{3}|x|]. \quad (1.9)$$

Consequently, by the standard random-walk analysis, we can readily show that the massless Gaussian two-point function $\langle \varphi_o \varphi_x \rangle_{\hat{\mathcal{J}}}^{\lambda=0}$ exhibits the asymptotic behavior

$$\langle \varphi_o \varphi_x \rangle_{\hat{\mathcal{J}}}^{\lambda=0} \underset{|x| \rightarrow \infty}{\sim} \frac{\frac{d}{2} \Gamma(\frac{d-2}{2}) \pi^{-d/2}}{\hat{\mathcal{J}} V |x|^{d-2}}. \quad (1.10)$$

For the φ^4 model with $\lambda > 0$, however, the last term on the left-hand side destroys linearity, and therefore it is not obvious any more whether we can get an explicit expression for the two-point function, or at least we can estimate its asymptotic behavior.

1.2 The main result

In this paper, we extend the lace-expansion methodology to factorize the nonlinear term in (1.6) and prove the expected asymptotic behavior of the critical two-point function for $d > 4$. By virtue of this approach, we can avoid the assumption of reflection positivity. The precise statement is the following.

Theorem 1.1. *For $d > 4$, there is a $\lambda_0 = \lambda_0(d, \mathcal{J}) \in (0, \infty)$ such that the following holds for all $\lambda \in [0, \lambda_0]$: there is a $\Phi_\mu(x) = \langle \varphi_o^2 \rangle_\mu \delta_{o,x} + O(\lambda)/(|x| \vee 1)^{3(d-2)}$ uniformly in $\mu > \mu_c$ such that a linearized version of the Schwinger-Dyson equation*

$$-\sum_v \mathcal{J}_{o,v} \langle \varphi_v \varphi_x \rangle_\mu + \mu \langle \varphi_o \varphi_x \rangle_\mu + \frac{\lambda}{2} \sum_v \Phi_\mu(v) \langle \varphi_v \varphi_x \rangle_\mu = \delta_{o,x} \quad (1.11)$$

holds. Consequently,

$$\mu_c = \hat{\mathcal{J}} - \frac{\lambda}{2} \langle \varphi_o^2 \rangle_{\mu_c} + O(\lambda^2), \quad (1.12)$$

and there are $A = \hat{\mathcal{J}}V + O(\lambda^2)$ and $\kappa > 2$ such that, as $|x| \rightarrow \infty$,

$$\langle \varphi_o \varphi_x \rangle_{\mu_c} = \frac{\frac{d}{2} \Gamma(\frac{d-2}{2}) / \pi^{d/2}}{A|x|^{d-2}} + O(|x|^{\kappa-d}). \quad (1.13)$$

Remark 1.2. (a) We may prove similar results for arbitrarily large λ if $\hat{\mathcal{J}}$ is sufficiently large (e.g., the nearest-neighbor model with $d \gg 4$). In fact, the $O(\lambda)$ term in the above Φ_μ is actually $O(\lambda/\mu^3)$ (cf., (3.93)). Although the constant in the $O(\lambda/\mu^3)$ term may depend on the range of \mathcal{J} , which is potentially large, the denominator μ^3 ($\simeq \hat{\mathcal{J}}^3$ around the critical point) should be large enough to cancel that effect.

(b) We may also extend the results to the case of power-law decaying spin-spin coupling, $\mathcal{J}(x) \propto |x|^{-d-\alpha}$ for some $\alpha > 0$. However, the variance V in (1.7) does not exist if $\alpha < 2$. In this case, the underlying random walk is in the domain of attraction of α -stable motion, and the critical two-point function $\langle \varphi_o \varphi_x \rangle_{\mu_c}$ should asymptotically be a multiple of $|x|^{\alpha-d}$ as $|x| \rightarrow \infty$, in dimensions $d > 2\alpha$. See [5] for more details.

(c) The actual proof of (1.12)–(1.13) assuming (1.11) goes as follows. First, we note that the sum of (1.11) yields

$$-\hat{\mathcal{J}} + \mu + \frac{\lambda}{2} \sum_v \Phi_\mu(v) = \chi_\mu^{-1}. \quad (1.14)$$

Using this, we can rearrange (1.11) as

$$\begin{aligned} \mu \langle \varphi_o \varphi_x \rangle_\mu &= \delta_{o,x} + \sum_v \frac{\mathcal{J}(v) - \frac{\lambda}{2} \Phi_\mu(v)}{\mu} \mu \langle \varphi_v \varphi_x \rangle_\mu \\ &= \delta_{o,x} + \sum_v \frac{\mu - \chi_\mu^{-1}}{\mu} \frac{\mathcal{J}(v) - \frac{\lambda}{2} \Phi_\mu(v)}{\hat{\mathcal{J}} - \frac{\lambda}{2} \sum_y \Phi_\mu(y)} \mu \langle \varphi_v \varphi_x \rangle_\mu. \end{aligned} \quad (1.15)$$

Let

$$\mathcal{G}_\mu(x) = \mu \langle \varphi_o \varphi_x \rangle_\mu, \quad \mathcal{D}_\mu(x) = \frac{\mathcal{J}(x) - \frac{\lambda}{2} \Phi_\mu(x)}{\hat{\mathcal{J}} - \frac{\lambda}{2} \sum_y \Phi_\mu(y)}, \quad (1.16)$$

so that

$$\mathcal{G}_\mu(x) = \delta_{o,x} + \sum_v \left(1 - \frac{\chi_\mu^{-1}}{\mu}\right) \mathcal{D}_\mu(v) \mathcal{G}_\mu(x-v). \quad (1.17)$$

This is identical to the convolution equation for the random-walk's Green function generated by the 1-step distribution \mathcal{D}_μ with killing rate χ_μ^{-1}/μ . Therefore, by the standard random-walk analysis (e.g., [5, Proposition 2.1]), we obtain

$$\mathcal{G}_{\mu_c}(x) = \frac{\frac{d}{2} \Gamma(\frac{d-2}{2}) / \pi^{d/2}}{\sum_y |y|^2 \mathcal{D}_{\mu_c}(y)} |x|^{2-d} + O(|x|^{\kappa-d}), \quad (1.18)$$

for some $\kappa > 2$, where \mathcal{D}_{μ_c} is defined in terms of arbitrary subsequential limit $\Phi_{\mu_c} \equiv \lim_{\mu_j \downarrow \mu_c} \Phi_{\mu_j}$, which exists and obeys

$$\Phi_{\mu_c}(x) = \langle \varphi_o^2 \rangle_{\mu_c} \delta_{o,x} + \frac{O(\lambda)}{(|x| \vee 1)^{3(d-2)}}, \quad (1.19)$$

due to the uniformity of Φ_μ in $\mu > \mu_c$. Using this and (1.14), we obtain

$$\mu_c = \hat{\mathcal{J}} - \frac{\lambda}{2} \sum_v \Phi_{\mu_c}(v) = \hat{\mathcal{J}} - \frac{\lambda}{2} \langle \varphi_o^2 \rangle_{\mu_c} + O(\lambda^2). \quad (1.20)$$

Moreover, since $3(d-2) = d+2+2(d-4)$, we obtain

$$\begin{aligned} \sum_y |y|^2 \mathcal{D}_{\mu_c}(y) &= \frac{1}{\hat{\mathcal{J}} - \frac{\lambda}{2} \sum_y \Phi_{\mu_c}(y)} \sum_{y \neq o} |y|^2 \left(\mathcal{J}(y) - \frac{\lambda}{2} \Phi_{\mu_c}(y) \right) \\ &= \frac{1}{\mu_c} \left(\hat{\mathcal{J}} V + \sum_{y \neq o} \frac{O(\lambda^2)}{|y|^{d+2(d-4)}} \right) \equiv \frac{A}{\mu_c}. \end{aligned} \quad (1.21)$$

This together with (1.18) implies (1.13).

1.3 Organization

The rest of this paper is organized as follows. In Section 2, as a preliminary section, we introduce some notation and summarize relevant properties of the two-point function. Then, in Section 3, we use those properties and the lace expansion for the Ising model to prove the main theorem, Theorem 1.1.

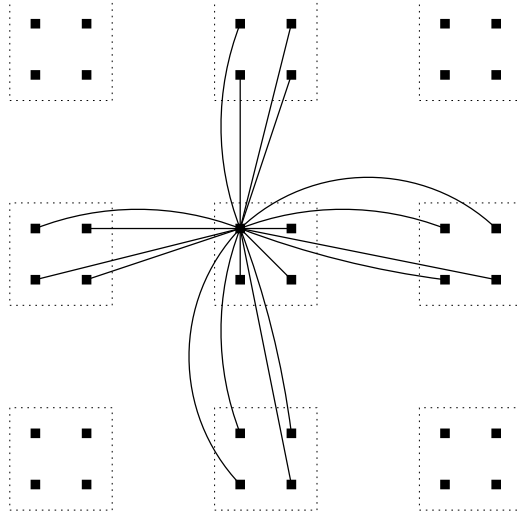


Figure 2: The nearest-neighbor bonds from a single vertex on $\tilde{\mathbb{Z}}_4^d \equiv \mathbb{Z}^d \times [4]$. Each dotted box collects four vertices with a common spatial coordinate.

2 Approximation by the Ising model

In this section, we briefly review two key components for the proof of Theorem 1.1. One of them is the Griffiths-Simon construction (Section 2.1), by which we can approximate the φ^4 model with a sum of N Ising systems. The other component is the random-current representation (Section 2.2), by which we can think of the Ising two-point function as a certain connectivity function. As a result, we can find many useful properties of the two-point function (Section 2.3). The lace expansion for the Ising model (Section 3.1) is one of them.

2.1 The Griffiths-Simon construction

Let

$$[N] = \{1, 2, \dots, N\}, \quad \tilde{\Lambda}_N = \Lambda \times [N], \quad (2.1)$$

and define the Ising Hamiltonian on $\tilde{\Lambda}_N$ as

$$\begin{aligned} H_{\tilde{\Lambda}_N}(\boldsymbol{\sigma}) &= -\frac{1}{2} \sum_{\substack{x \neq y \in \Lambda \\ i, j \in [N]}} J_{x,y} \sigma_{x,i} \sigma_{y,j} - \frac{I}{2} \sum_{\substack{x \in \Lambda \\ i, j \in [N]}} \sigma_{x,i} \sigma_{x,j} \\ &= -\frac{1}{2} \sum_{x \neq y \in \Lambda} J_{x,y} \tilde{\sigma}_x \tilde{\sigma}_y - \frac{I}{2} \sum_{x \in \Lambda} \tilde{\sigma}_x^2, \end{aligned} \quad (2.2)$$

where $\boldsymbol{\sigma} \equiv (\sigma_{(x,i)})_{(x,i) \in \Lambda \times [N]}$ is the Ising-spin configuration and

$$\tilde{\sigma}_x \equiv \sum_{i \in [N]} \sigma_{(x,i)} \quad (2.3)$$

is what we call in this paper the block spin at $x \in \Lambda$. It is known [21] that, if I, J and σ are determined from $\lambda, \mu, \mathcal{J}$ and φ with proper scaling (e.g., $\tilde{\sigma}_x \mapsto \epsilon_N \tilde{\sigma}_x$, with an appropriate scaling factor ϵ_N), then

$$\epsilon_N^2 \langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N} \equiv \epsilon_N^2 \frac{\sum_{\sigma \in \{\pm 1\}^{\tilde{\Lambda}_N}} \tilde{\sigma}_o \tilde{\sigma}_x e^{-H_{\tilde{\Lambda}_N}(\sigma)}}{\sum_{\sigma \in \{\pm 1\}^{\tilde{\Lambda}}} e^{-H_{\tilde{\Lambda}_N}(\sigma)}} \xrightarrow{N \uparrow \infty} \frac{\int_{\mathbb{R}^\Lambda} \varphi_o \varphi_x e^{-\mathcal{H}_\Lambda(\varphi)} d\varphi}{\int_{\mathbb{R}^\Lambda} e^{-\mathcal{H}_\Lambda(\varphi)} d\varphi} = \langle \varphi_o \varphi_x \rangle_{\mu, \Lambda}. \quad (2.4)$$

Now, we provide heuristic explanation of the aforementioned proper scaling. For more details, refer to [21]. First, we note that the marginal distribution given $\tilde{\sigma} = (\tilde{\sigma}_x)_{x \in \Lambda}$ is

$$\begin{aligned} & \left(\frac{1}{2}\right)^{|\tilde{\Lambda}_N|} \sum_{\substack{\sigma \in \{\pm 1\}^{\tilde{\Lambda}_N} \\ (\tilde{\sigma} \text{ fixed})}} e^{-H_{\tilde{\Lambda}_N}(\sigma)} \\ &= \exp \left(\frac{1}{2} \sum_{x \neq y \in \Lambda} J_{x,y} \tilde{\sigma}_x \tilde{\sigma}_y + \frac{I}{2} \sum_{x \in \Lambda} \tilde{\sigma}_x^2 \right) \prod_{x \in \Lambda} \sum_{\substack{\sigma_{(x,1)}, \dots, \sigma_{(x,N)} \\ (\tilde{\sigma}_x \text{ fixed})}} \left(\frac{1}{2}\right)^N \\ &= \exp \left(\frac{1}{2} \sum_{x \neq y \in \Lambda} J_{x,y} \tilde{\sigma}_x \tilde{\sigma}_y + \frac{I}{2} \sum_{x \in \Lambda} \tilde{\sigma}_x^2 \right) \prod_{x \in \Lambda} \binom{N}{\frac{N+\tilde{\sigma}_x}{2}} \left(\frac{1}{2}\right)^N. \end{aligned} \quad (2.5)$$

By Stirling's formula (i.e., $\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$ for all $n \in \mathbb{N}$),

$$\begin{aligned} & \log \left(\binom{N}{\frac{N+\tilde{\sigma}_x}{2}} \left(\frac{1}{2}\right)^N \right) \\ &= -\frac{N}{2} \left(\left(1 + \frac{\tilde{\sigma}_x}{N}\right) \log \left(1 + \frac{\tilde{\sigma}_x}{N}\right) + \left(1 - \frac{\tilde{\sigma}_x}{N}\right) \log \left(1 - \frac{\tilde{\sigma}_x}{N}\right) \right) + O(\log N). \end{aligned} \quad (2.6)$$

Let $\eta_x = \tilde{\sigma}_x/N$. Then, by the Taylor expansion,

$$(1 \pm \eta_x) \log(1 \pm \eta_x) = (1 \pm \eta_x) \left(\pm \eta_x - \frac{\eta_x^2}{2} \pm \frac{\eta_x^3}{3} - \frac{\eta_x^4}{4} \pm \frac{\eta_x^5}{5} + o(\eta_x^5) \right), \quad (2.7)$$

which implies

$$(1 + \eta_x) \log(1 + \eta_x) + (1 - \eta_x) \log(1 - \eta_x) = 2 \left(\frac{\eta_x^2}{2} + \frac{\eta_x^4}{12} + o(\eta_x^5) \right). \quad (2.8)$$

Therefore,

$$(2.5) \propto \exp \left(\frac{1}{2} \sum_{x \neq y \in \Lambda} J_{x,y} \tilde{\sigma}_x \tilde{\sigma}_y \right) \prod_{x \in \Lambda} \exp \left(\frac{1}{2} \left(I - \frac{1}{N} \right) \tilde{\sigma}_x^2 - \frac{1}{12} \frac{\tilde{\sigma}_x^4}{N^3} + o\left(\frac{\tilde{\sigma}_x^5}{N^4}\right) \right). \quad (2.9)$$

Let

$$\frac{1}{12} \frac{\tilde{\sigma}_x^4}{N^3} = \frac{\lambda}{4!} \varphi_x^4, \quad \text{or equivalently} \quad \varphi_x = \epsilon_N \tilde{\sigma}_x \equiv \left(\frac{\lambda}{2} N^3 \right)^{-1/4} \tilde{\sigma}_x, \quad (2.10)$$

and

$$J_{x,y} = \mathcal{J}_{x,y} \epsilon_N^2, \quad I = \frac{1}{N} - \mu \epsilon_N^2. \quad (2.11)$$

Then, we arrive at

$$(2.9) = \exp \left(\frac{1}{2} \sum_{x \neq y \in \Lambda} \mathcal{J}_{x,y} \varphi_x \varphi_y - \sum_{x \in \Lambda} \left(\frac{\mu}{2} \varphi_x^2 + \frac{\lambda}{4!} \varphi_x^4 + o(N^{-1/4} \varphi_x^5) \right) \right) \sim e^{-\mathcal{H}_\Lambda(\varphi)}. \quad (2.12)$$

In Section 3, we apply the lace expansion [20] to the ferromagnetic Ising model defined by the Hamiltonian (2.2). For the ferromagnetic condition $I \geq 0$, we assume from now on

$$N \geq \frac{2\mu^2}{\lambda}. \quad (2.13)$$

2.2 The random-current representation

In this subsection, we explain the random-current representation (e.g., [1]) and introduce some notation.

First, we rewrite the Ising Hamiltonian (2.2) as

$$\begin{aligned} H_{\tilde{\Lambda}_N}(\boldsymbol{\sigma}) &= -\frac{1}{2} \sum_{\substack{x \neq y \in \Lambda \\ i,j \in [N]}} J_{x,y} \sigma_{(x,i)} \sigma_{(y,j)} - \frac{I}{2} \sum_{\substack{x \in \Lambda \\ i \neq j \in [N]}} \sigma_{(x,i)} \sigma_{(x,j)} - \frac{I}{2} |\tilde{\Lambda}_N| \\ &= - \sum_{b \in \mathbb{B}_{\tilde{\Lambda}_N}} \tilde{J}_b \sigma_{b_1} \sigma_{b_2} - \frac{I}{2} |\tilde{\Lambda}|, \end{aligned} \quad (2.14)$$

where $\mathbb{B}_{\tilde{\Lambda}_N} = \{b = \{b_1, b_2\} : b_1 \neq b_2 \in \tilde{\Lambda}_N\}$ and

$$\tilde{J}_{(x,i),(y,j)} = J_{x,y} + I \delta_{x,y} (1 - \delta_{i,j}) = \begin{cases} J_{x,y} & [x \neq y], \\ I & [x = y \text{ and } i \neq j], \\ 0 & [\text{otherwise}]. \end{cases} \quad (2.15)$$

Then, by expanding exponentials, we obtain

$$\begin{aligned} \left(\frac{1}{2} \right)^{|\tilde{\Lambda}_N|} \sum_{\boldsymbol{\sigma} \in \{\pm 1\}^{\tilde{\Lambda}}} e^{-H_{\tilde{\Lambda}_N}(\boldsymbol{\sigma})} &= \left(\frac{e^{I/2}}{2} \right)^{|\tilde{\Lambda}_N|} \sum_{\boldsymbol{\sigma} \in \{\pm 1\}^{\tilde{\Lambda}_N}} \prod_{b \in \mathbb{B}_{\tilde{\Lambda}_N}} \sum_{n_b \in \mathbb{Z}_+} \frac{(\tilde{J}_b \sigma_{b_1} \sigma_{b_2})^{n_b}}{n_b!} \\ &= e^{I|\tilde{\Lambda}_N|/2} \sum_{\mathbf{n}=(n_b)} \underbrace{\left(\prod_{b \in \mathbb{B}_{\tilde{\Lambda}_N}} \frac{\tilde{J}_b^{n_b}}{n_b!} \right)}_{w_{\tilde{\Lambda}_N}(\mathbf{n})} \prod_{\tilde{x} \in \tilde{\Lambda}_N} \underbrace{\frac{1}{2} \sum_{\sigma_{\tilde{x}} \in \{\pm 1\}} \sigma_{\tilde{x}}^{\sum_{b \ni \tilde{x}} n_b}}_{\mathbb{1}_{\{\sum_{b \ni \tilde{x}} n_b \text{ is even}\}}} \\ &= e^{I|\tilde{\Lambda}_N|/2} \sum_{\partial \mathbf{n} = \emptyset} w_{\tilde{\Lambda}_N}(\mathbf{n}), \end{aligned} \quad (2.16)$$

where $\partial \mathbf{n} \equiv \{\tilde{x} \in \tilde{\Lambda}_N : \sum_{b \ni \tilde{x}} n_b \text{ is odd}\}$ is the set of sources in the current configuration $\mathbf{n} = (n_b) \in \mathbb{Z}_+^{\tilde{\Lambda}_N}$. Similarly, for $\tilde{x}, \tilde{y} \in \tilde{\Lambda}_N$,

$$\left(\frac{1}{2}\right)^{|\tilde{\Lambda}_N|} \sum_{\sigma \in \{\pm 1\}^{\tilde{\Lambda}}} \sigma_{\tilde{x}} \sigma_{\tilde{y}} e^{-H_{\tilde{\Lambda}_N}(\sigma)} = e^{I|\tilde{\Lambda}_N|/2} \sum_{\partial \mathbf{n} = \tilde{x} \Delta \tilde{y}} w_{\tilde{\Lambda}}(\mathbf{n}), \quad (2.17)$$

where $\tilde{x} \Delta \tilde{y}$ is the abbreviation for the symmetric difference $\{\tilde{x}\} \Delta \{\tilde{y}\}$. As a result, we arrive at the random-current representation for the Ising two-point function

$$\langle\langle \sigma_{\tilde{x}} \sigma_{\tilde{y}} \rangle\rangle_{\tilde{\Lambda}_N} \equiv \frac{\sum_{\sigma \in \{\pm 1\}^{\tilde{\Lambda}_N}} \sigma_{\tilde{x}} \sigma_{\tilde{y}} e^{-H_{\tilde{\Lambda}_N}(\sigma)}}{\sum_{\sigma \in \{\pm 1\}^{\tilde{\Lambda}}} e^{-H_{\tilde{\Lambda}_N}(\sigma)}} = \frac{\sum_{\partial \mathbf{n} = \tilde{x} \Delta \tilde{y}} w_{\tilde{\Lambda}_N}(\mathbf{n})}{\sum_{\partial \mathbf{n} = \emptyset} w_{\tilde{\Lambda}}(\mathbf{n})}. \quad (2.18)$$

2.3 Basic properties of the two-point function

In this subsection, we summarize the properties of the Ising two-point function obtained from the random-current representation (2.18).

Lemma 2.1 ((2.28) and (2.37) of [5]). *Let $\Lambda \subset \mathbb{Z}^d$ be the d -dimensional hypercube centered at the origin $o \in \mathbb{Z}^d$. For any $I \geq 0$, the following two inequalities hold:*

(i) *For any $x \in \Lambda$,*

$$\langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} - \delta_{\tilde{o}, \tilde{x}} \leq \sum_{\tilde{v} \in \tilde{\Lambda}_N} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N}. \quad (2.19)$$

(ii) *Suppose that the radius of Λ is bigger than a given $\ell < \infty$. Then, for $|x| > \ell$,*

$$\langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} \leq \sum_{\substack{\tilde{u}, \tilde{v} \in \tilde{\Lambda}_N \\ (|u| \leq \ell < |v|)}} \langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle\rangle_{\tilde{\Lambda}_N} (\tanh J_{u,v}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N}. \quad (2.20)$$

Proposition 2.2. *Let*

$$G_{\tilde{\Lambda}_N}(o, x) = \frac{1 - (N - 1) \tanh I}{N} \langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N}, \quad (2.21)$$

and denote its (unique and translation-invariant) infinite-volume limit by

$$G_N(x) = \lim_{\Lambda \uparrow \mathbb{Z}^d} G_{\tilde{\Lambda}_N}(o, x). \quad (2.22)$$

Let $\mu > \mu_N \equiv \inf\{\mu : \sum_x G_N(x) < \infty\}$.

(i) *Let $S_p(x)$ be the random-walk's Green function whose 1-step distribution and fugacity are defined, respectively, as*

$$D(v) = \frac{\tanh J_{o,v}}{\sum_{v \in \mathbb{Z}^d} \tanh J_{o,v}}, \quad p = \frac{N \sum_{v \in \mathbb{Z}^d} \tanh J_{o,v}}{1 - (N - 1) \tanh I}. \quad (2.23)$$

Then, $G_N(x) \leq S_p(x)$ for all $x \in \mathbb{Z}^d$.

(ii) Suppose that there is an $\alpha > 0$ such that $J_{o,x} = O(|x|^{-d-\alpha})$ as $|x| \rightarrow \infty$ (α can be an arbitrarily large number in the current setting). Then, there is a $K_\mu < \infty$ such that $G_N(x) \leq K_\mu(|x| \vee 1)^{-d-\alpha}$ for all $x \in \mathbb{Z}^d$.

Proof of (i). Let $\tilde{o} = (o, i)$ and $\tilde{x} = (x, j)$. Summing (2.19) over $i, j \in [N]$, we obtain

$$\begin{aligned} \langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N} - N\delta_{o,x} &\leq \sum_{\tilde{v} \in \tilde{\Lambda}_N} \sum_{i \in [N]} (\tanh \tilde{J}_{(o,i),\tilde{v}}) \langle\langle \sigma_{\tilde{v}} \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N} \\ &= \sum_{v \in \Lambda} \sum_{i,j \in [N]} \tanh \tilde{J}_{(o,i),(v,j)} \frac{\langle\langle \tilde{\sigma}_v \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N}}{N} \\ &\leq (N-1)(\tanh I) \langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N} + N \sum_{v \in \Lambda} (\tanh J_{o,v}) \langle\langle \tilde{\sigma}_v \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N}. \end{aligned} \quad (2.24)$$

Solving this inequality for $\langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N}$ and using (2.21)–(2.23), we arrive at

$$G_N(x) \leq \delta_{o,x} + p(D * G_N)(x) \equiv \delta_{o,x} + p \sum_{v \in \mathbb{Z}^d} D(v) G_N(x-v). \quad (2.25)$$

Repeated application of this inequality yields $G_N(x) \leq \sum_{n=0}^{\infty} p^n D^{*n}(x) = S_p(x)$. \blacksquare

Proof of (ii). Let $\tilde{o} = (o, i)$ and $\tilde{x} = (x, j)$. Summing (2.20) over $i, j \in [N]$ and using (2.21)–(2.23), we readily obtain the Simon-Lieb type inequality

$$\begin{aligned} G_N(x) &\leq \sum_{\substack{u,v \in \mathbb{Z}^d \\ (|u| \leq \ell < |v|)}} G_N(u) \frac{N \tanh J_{u,v}}{1 - (N-1) \tanh I} G_N(x-v) \\ &= \sum_{\substack{u,v \in \mathbb{Z}^d \\ (|u| \leq \ell < |v|)}} G_N(u) p D(v-u) G_N(x-v). \end{aligned} \quad (2.26)$$

Under the assumption on the decay of J , the 1-step distribution D obeys the same asymptotic bound $D(x) = O(|x|^{-d-\alpha})$ as $|x| \rightarrow \infty$. Then, we can follow the same proof as [5, Lemma 2.4] to obtain $G_N(x) \leq K_\mu(|x| \vee 1)^{-d-\alpha}$, where K_μ is finite as long as $\mu > \mu_N$. \blacksquare

Before closing this section, we provide bounds on $\langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N}$ in terms of $G_N(x)$. Notice that, by symmetry,

$$\langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N} = N \langle\langle \sigma_{\tilde{o}} \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N} = N \times \begin{cases} 1 + (N-1) \langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{o}'} \rangle\rangle_{\tilde{\Lambda}_N} & [x = o], \\ N \langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} & [x \neq o], \end{cases} \quad (2.27)$$

where \tilde{o}' is another vertex than \tilde{o} whose spatial coordinate is o . Then, for $\tilde{x} \neq \tilde{o}$,

$$\begin{aligned} \langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} &= \frac{\frac{1}{N} \langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\Lambda}_N} - \delta_{o,x}}{N - \delta_{o,x}} \leq \frac{G_{\tilde{\Lambda}_N}(o, x)}{(1 - (N-1) \tanh I)(N-1)} \\ &\leq \frac{G_{\tilde{\Lambda}_N}(o, x)}{\mu \epsilon_N^2 (N-1)^2}. \end{aligned} \quad (2.28)$$

Since $N^2/(N-1)^2 \leq 4$ for $N \geq 2$ and $G_{\tilde{\Lambda}_N}(o, x) \leq G_N(x)$, we have

$$\langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} \leq \delta_{\tilde{o}, \tilde{x}} + (1 - \delta_{\tilde{o}, \tilde{x}}) \frac{4G_N(x)}{\mu \epsilon_N^2 N^2}. \quad (2.29)$$

Similarly, we obtain

$$\begin{aligned} \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} &\leq (\tanh I) \langle\langle \tilde{\sigma}_o \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} + \sum_v (\tanh J_{o, v}) \langle\langle \tilde{\sigma}_v \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} \\ &= \frac{(\tanh I) G_{\tilde{\Lambda}_N}(o, x) + \sum_v (\tanh J_{o, v}) G_{\tilde{\Lambda}_N}(v, x)}{1 - (N-1) \tanh I} \\ &\leq \frac{2}{\mu \epsilon_N^2 N^2} \left(G_N(x) + \hat{\mathcal{J}} \epsilon_N^2 N (D * G_N)(x) \right), \end{aligned} \quad (2.30)$$

where the factor $\hat{\mathcal{J}} \epsilon_N^2 N$ is smaller than 1 if we choose $N > 2 \hat{\mathcal{J}}^2 / \lambda$.

3 Proof of the main theorem

In this section, we prove Theorem 1.1 by first showing the expected x -space infrared bound (Section 3.2), which has been proven to be true only for the nearest-neighbor model so far [23]. Then, by using that infrared bound, we derive the linear Schwinger-Dyson equation (Section 3.3), which is the core of the main theorem. Both sections depend heavily on the lace expansion for the Ising model (Section 3.1).

3.1 The lace expansion

The lace expansion has been successful in proving asymptotic behavior of the critical two-point function for various models. In particular, for the ferromagnetic Ising model, which is considered to be in the same universality class as the φ^4 model, the critical two-point function is proven to be $|x|^{2-d}$ times a model-dependent constant as $|x| \rightarrow \infty$ when the spin-spin coupling has a finite $(2 + \varepsilon)$ th moment for some $\varepsilon > 0$ [5, 20].

In this subsection, we apply the lace expansion for the Ising model [20] to the approximate model constructed in Section 2.1 and investigate the N -dependence of the expansion coefficients.

According to [20], for every $T \in \mathbb{Z}_+$, there are functions $\pi_{\tilde{\Lambda}_N}^{(\leq T)}$ and $r_{\tilde{\Lambda}_N}^{(T+1)}$ on $\tilde{\Lambda}_N \times \tilde{\Lambda}_N$ such that the following identity holds:

$$\langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} = \pi_{\tilde{\Lambda}_N}^{(\leq T)}(\tilde{o}, \tilde{x}) + \sum_{\tilde{u}, \tilde{v} \in \tilde{\Lambda}_N} \pi_{\tilde{\Lambda}_N}^{(\leq T)}(\tilde{o}, \tilde{u}) (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} + r_{\tilde{\Lambda}_N}^{(T+1)}(\tilde{o}, \tilde{x}). \quad (3.1)$$

In fact, $\pi_{\tilde{\Lambda}_N}^{(\leq T)}(\tilde{o}, \tilde{x})$ is an alternating sum of nonnegative functions $\pi_{\tilde{\Lambda}_N}^{(t)}(\tilde{o}, \tilde{x})$, $0 \leq t \leq T$. Moreover, the remainder $r_{\tilde{\Lambda}_N}^{(T+1)}(\tilde{o}, \tilde{x})$ is bounded uniformly in x as

$$\begin{aligned} |r_{\tilde{\Lambda}_N}^{(T+1)}(\tilde{o}, \tilde{x})| &\leq \sum_{\tilde{u}} \pi_{\tilde{\Lambda}_N}^{(T)}(\tilde{o}, \tilde{u}) \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} \\ &\leq (1 + \hat{\mathcal{J}} \epsilon_N^2 N) \sum_{\tilde{u}} \pi_{\tilde{\Lambda}_N}^{(T)}(\tilde{o}, \tilde{u}), \end{aligned} \quad (3.2)$$

$$\begin{aligned}
\pi_{\tilde{\Lambda}_N}^{(0)}(\tilde{o}, \tilde{x}) &\leq \tilde{o} \text{---} \text{---} \tilde{x} & \pi_{\tilde{\Lambda}_N}^{(1)}(\tilde{o}, \tilde{x}) &\leq \tilde{o} \text{---} \text{---} \tilde{x} + \dots \\
\pi_{\tilde{\Lambda}_N}^{(2)}(\tilde{o}, \tilde{x}) &\leq \tilde{o} \text{---} \text{---} \tilde{x} + \tilde{o} \text{---} \text{---} \tilde{x} + \dots
\end{aligned}$$

Figure 3: The leading bounding diagrams for $\pi_{\tilde{\Lambda}_N}^{(t)}(\tilde{o}, \tilde{x})$, $t = 0, 1, 2$. Each line segment represents an Ising two-point function, e.g., $\pi_{\tilde{\Lambda}_N}^{(0)}(\tilde{o}, \tilde{x}) \leq \langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N}^3$. The unlabeled vertices are summer over $\tilde{\Lambda}_N$. The tiny rectangles represent $\tanh \tilde{J}$.

where we have used the inequality

$$\sum_{\tilde{v}} \tanh \tilde{J}_{\tilde{u}, \tilde{v}} = (N-1) \tanh I + N \sum_v \tanh J_{u,v} \leq 1 + \hat{\mathcal{J}} \epsilon_N^2 N. \quad (3.3)$$

The functions $\pi_{\tilde{\Lambda}_N}^{(t)}(\tilde{o}, \tilde{x})$, $t \geq 0$, are defined by using the random-current representation. For example,

$$\pi_{\tilde{\Lambda}_N}^{(0)}(\tilde{o}, \tilde{x}) = \frac{\sum_{\partial \mathbf{n} = \tilde{o} \Delta \tilde{x}} w_{\tilde{\Lambda}_N}(\mathbf{n}) \mathbb{1}_{\{\tilde{o} \longleftrightarrow \tilde{x}\}}}{\sum_{\partial \mathbf{n} = \emptyset} w_{\tilde{\Lambda}_N}(\mathbf{n})}, \quad (3.4)$$

where $\tilde{o} \longleftrightarrow \tilde{x}$ means that there are at least two bond-disjoint paths in $\tilde{\Lambda}_N$ from \tilde{o} to \tilde{x} , consisting of bonds b with $n_b > 0$. The precise definitions of those functions are irrelevant, and we refrain from showing them here. What matters most is their diagrammatic bounds [20, Proposition 4.1] (cf., Figure 3). Combining with (2.29)–(2.30), we can show the following proposition.

Proposition 3.1. *Let*

$$\|x\| = |x| \vee 1, \quad \tilde{\lambda} = \frac{1}{\mu \epsilon_N^2 N^2} = \frac{1}{\mu} \sqrt{\frac{\lambda}{2N}}, \quad (3.5)$$

and let $\lambda \ll 2\mu^2$ and $N > 2(\hat{\mathcal{J}} \vee \mu)^2/\lambda$ (so that $\hat{\mathcal{J}} \epsilon_N^2 N < 1$ and $\tilde{\lambda} < \tilde{\lambda}^2 N \ll 1$). Suppose that $\sup_x \|x\|^{d-2} G_N(x)$ is bounded by a constant which is independent of λ, μ, N . Then, for $d > 4$,

$$0 \leq \pi_{\tilde{\Lambda}_N}^{(0)}(\tilde{o}, \tilde{x}) - \delta_{\tilde{o}, \tilde{x}} \leq \frac{O(\tilde{\lambda})^3}{\|x\|^{3(d-2)}}, \quad (3.6)$$

$$0 \leq \pi_{\tilde{\Lambda}_N}^{(1)}(\tilde{o}, \tilde{x}) - \delta_{\tilde{o}, \tilde{x}} \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}} \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{o}} \rangle\rangle_{\tilde{\Lambda}_N}) \leq O(\tilde{\lambda})^2 \left(\delta_{\tilde{o}, \tilde{x}} + \frac{O(\tilde{\lambda})}{\|x\|^{3(d-2)}} \right), \quad (3.7)$$

$$0 \leq \pi_{\tilde{\Lambda}_N}^{(t)}(\tilde{o}, \tilde{x}) \leq O(\tilde{\lambda})^t \left(\delta_{\tilde{o}, \tilde{x}} + \frac{O(\tilde{\lambda})}{\|x\|^{3(d-2)}} \right) \quad [t \geq 2], \quad (3.8)$$

where the constants in the $O(\tilde{\lambda})$ terms are independent of λ, μ, N and Λ .

As a result of the above proposition and (3.2), we have $\lim_{T \rightarrow \infty} r_{\tilde{\Lambda}_N}^{(T+1)}(\tilde{o}, \tilde{x}) = 0$, and therefore the alternating series $\pi_{\tilde{\Lambda}_N}(\tilde{o}, \tilde{x}) \equiv \lim_{T \rightarrow \infty} \pi_{\tilde{\Lambda}_N}^{(\leq T)}(\tilde{o}, \tilde{x})$ converges and satisfies

$$\left| \pi_{\tilde{\Lambda}_N}(\tilde{o}, \tilde{x}) - \delta_{\tilde{o}, \tilde{x}} \left(1 - \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \langle \langle \sigma_{\tilde{v}} \sigma_{\tilde{o}} \rangle \rangle_{\tilde{\Lambda}_N} \right) \right| \leq O(\tilde{\lambda}^2) \left(\delta_{\tilde{o}, \tilde{x}} + \frac{O(\tilde{\lambda})}{\|x\|^{3(d-2)}} \right). \quad (3.9)$$

We will use this estimate in the next subsection to investigate (the $T \rightarrow \infty$ limit of) (3.1) and prove that the assumed bound on G_N in Proposition 3.1 indeed holds.

Proof of Proposition 3.1. The inequality (3.6) is readily obtained by applying (2.29) to the diagrammatic bound $\pi_{\tilde{\Lambda}_N}^{(0)}(\tilde{o}, \tilde{x}) \leq \langle \langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle \rangle_{\tilde{\Lambda}_N}^3$ in [20, Proposition 4.1]. The proof of the other inequalities (3.7)–(3.8) are much more involved, and we only explain in detail how to bound the leading diagram for $\pi_{\tilde{\Lambda}_N}^{(1)}(\tilde{o}, \tilde{x})$, which is depicted in Figure 3:

$$\begin{aligned} \pi_{\tilde{\Lambda}_N}^{(1)}(\tilde{o}, \tilde{x}) &\leq \sum_{\tilde{u}, \tilde{v}, \tilde{w}, \tilde{y}} \langle \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle \rangle_{\tilde{\Lambda}_N}^2 \langle \langle \sigma_{\tilde{o}} \sigma_{\tilde{w}} \rangle \rangle_{\tilde{\Lambda}_N} \langle \langle \sigma_{\tilde{w}} \sigma_{\tilde{u}} \rangle \rangle_{\tilde{\Lambda}_N} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle \langle \sigma_{\tilde{v}} \sigma_{\tilde{y}} \rangle \rangle_{\tilde{\Lambda}_N} \\ &\quad \times \langle \langle \sigma_{\tilde{y}} \sigma_{\tilde{w}} \rangle \rangle_{\tilde{\Lambda}_N} \langle \langle \sigma_{\tilde{w}} \sigma_{\tilde{x}} \rangle \rangle_{\tilde{\Lambda}_N} \langle \langle \sigma_{\tilde{y}} \sigma_{\tilde{x}} \rangle \rangle_{\tilde{\Lambda}_N}^2 + \text{error terms}. \end{aligned} \quad (3.10)$$

The error terms in the above inequality and the higher order functions $\pi_{\tilde{\Lambda}_N}^{(t)}(\tilde{o}, \tilde{x})$, $t \geq 2$, can be estimated similarly (see also [20, Section 5.1]), and we refrain from showing them here.

Now, we prove that the sum on the right-hand side of (3.10) obeys the inequality (3.7). First, we split it into three sums depending on whether (i) $\tilde{o} = \tilde{u} = \tilde{w} = \tilde{y} = \tilde{x}$, (ii) $\tilde{o} \neq \tilde{u}$ and $\tilde{w} = \tilde{y} = \tilde{x}$, or (iii) $\tilde{y} \neq \tilde{x}$. Then, we obtain

$$\begin{aligned} \text{The sum in (3.10)} &\leq \delta_{\tilde{o}, \tilde{x}} \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \langle \langle \sigma_{\tilde{v}} \sigma_{\tilde{o}} \rangle \rangle_{\tilde{\Lambda}_N} \\ &\quad + \langle \langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle \rangle_{\tilde{\Lambda}_N} \sum_{\tilde{u} (\neq \tilde{o}), \tilde{v}} \langle \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle \rangle_{\tilde{\Lambda}_N}^2 \langle \langle \sigma_{\tilde{x}} \sigma_{\tilde{u}} \rangle \rangle_{\tilde{\Lambda}_N} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle \langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle \rangle_{\tilde{\Lambda}_N} \\ &\quad + \sum_{\tilde{u}, \tilde{v}, \tilde{w}, \tilde{y} (\neq \tilde{x})} \langle \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle \rangle_{\tilde{\Lambda}_N}^2 \langle \langle \sigma_{\tilde{o}} \sigma_{\tilde{w}} \rangle \rangle_{\tilde{\Lambda}_N} \langle \langle \sigma_{\tilde{w}} \sigma_{\tilde{u}} \rangle \rangle_{\tilde{\Lambda}_N} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle \langle \sigma_{\tilde{v}} \sigma_{\tilde{y}} \rangle \rangle_{\tilde{\Lambda}_N} \\ &\quad \times \langle \langle \sigma_{\tilde{y}} \sigma_{\tilde{w}} \rangle \rangle_{\tilde{\Lambda}_N} \langle \langle \sigma_{\tilde{w}} \sigma_{\tilde{x}} \rangle \rangle_{\tilde{\Lambda}_N} \langle \langle \sigma_{\tilde{y}} \sigma_{\tilde{x}} \rangle \rangle_{\tilde{\Lambda}_N}^2. \end{aligned} \quad (3.11)$$

In fact, the first term on the right-hand side is the trivial contribution to $\pi_{\tilde{\Lambda}_N}^{(1)}(\tilde{o}, \tilde{x})$, and therefore $\pi_{\tilde{\Lambda}_N}^{(1)}(\tilde{o}, \tilde{x}) - \delta_{\tilde{o}, \tilde{x}} \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \langle \langle \sigma_{\tilde{v}} \sigma_{\tilde{o}} \rangle \rangle_{\tilde{\Lambda}_N} \geq 0$.

It remains to show that the second and third terms on the right-hand side of the above inequality are bounded by the right-hand side of (3.7). In order to achieve this goal, we use the following convolution bounds.

Lemma 3.2 ([5, 13]). *(i) For any $a \geq b > 0$ with $a \neq d$ and $a + b > d$, there is a $C < \infty$ such that*

$$\sum_{y \in \mathbb{Z}^d} \|x - y\|^{-a} \|y\|^{-b} \leq C \|x\|^{a \vee d - a - b}. \quad (3.12)$$

(ii) Let f and g be functions on \mathbb{Z}^d , with g being \mathbb{Z}^d -symmetric. Suppose that there are $C_1, C_2, C_3 > 0$ and $\rho > 0$ such that

$$f(x) = C_1 \|x\|^{2-d}, \quad |g(x)| \leq C_2 \delta_{o,x} + C_3 \|x\|^{-d-\rho}.$$

Then there is a $\rho' \in (0, \rho \wedge 2)$ such that, for $d > 2$,

$$(f * g)(x) = \frac{C_1 \|g\|_1}{\|x\|^{d-2}} + \frac{O(C_1 C_3)}{\|x\|^{d-2+\rho'}}. \quad (3.13)$$

We use Lemma 3.2(i) to control the sums over $\tilde{u}, \tilde{w}, \tilde{y} \in \Lambda_N$ in (3.10), which correspond to the unlabeled vertices of degree 4 in the bounding diagram in Figure 3. For example, by Lemma 3.2(i) with $a = b = d - 2$, we obtain that, for $d > 4$,

$$\begin{aligned} & \sum_{\tilde{v} \in \tilde{\Lambda}_N} \frac{\tilde{\lambda}}{\|x_1 - v\|^{d-2}} \frac{\tilde{\lambda}}{\|v - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - v\|^{d-2}} \frac{\tilde{\lambda}}{\|v - x_4\|^{d-2}} \\ &= N \sum_{v \in \mathbb{Z}^d} \frac{\tilde{\lambda}}{(\|x_1 - v\| \vee \|v - x_2\|)^{d-2}} \frac{\tilde{\lambda}}{(\|x_1 - v\| \wedge \|v - x_2\|)^{d-2}} \\ & \quad \times \frac{\tilde{\lambda}}{(\|x_3 - v\| \vee \|v - x_4\|)^{d-2}} \frac{\tilde{\lambda}}{(\|x_3 - v\| \wedge \|v - x_4\|)^{d-2}} \\ &\leq 4^{d-2} C \tilde{\lambda}^2 N \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}}, \end{aligned} \quad (3.14)$$

where we have used the triangle inequality $\|x_i - v\| \vee \|v - x_j\| \geq \|x_i - x_j\|/2$ (cf., Figure 4). Similarly, if Kronecker's delta is added to one of those four fractions, then we have

$$\begin{aligned} & \sum_{\tilde{v} \in \tilde{\Lambda}_N} \left(\delta_{\tilde{x}_1, \tilde{v}} + \frac{\tilde{\lambda}}{\|x_1 - v\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|v - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - v\|^{d-2}} \frac{\tilde{\lambda}}{\|v - x_4\|^{d-2}} \\ &\leq \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \frac{2^{d-2} \tilde{\lambda}^2}{\|x_3 - x_4\|^{d-2}} + 4^{d-2} C \tilde{\lambda}^2 N \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}} \\ &\leq C' \tilde{\lambda}^2 N \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}}, \end{aligned} \quad (3.15)$$

with $C' = 2^{d-2} + 4^{d-2} C$, where we have used the assumption $\tilde{\lambda} < \tilde{\lambda}^2 N$. Moreover, if there are two fractions with Kronecker's delta, then we have

$$\begin{aligned} & \sum_{\tilde{v} \in \tilde{\Lambda}_N} \left(\delta_{\tilde{x}_1, \tilde{v}} + \frac{\tilde{\lambda}}{\|x_1 - v\|^{d-2}} \right) \left(\delta_{\tilde{v}, \tilde{x}_2} + \frac{\tilde{\lambda}}{\|v - x_2\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|x_3 - v\|^{d-2}} \frac{\tilde{\lambda}}{\|v - x_4\|^{d-2}} \\ &\leq \left(\delta_{\tilde{x}_1, \tilde{x}_2} + \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \right) \frac{2^{d-2} \tilde{\lambda}^2}{\|x_3 - x_4\|^{d-2}} + C' \tilde{\lambda}^2 N \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}} \\ &\leq (2^{d-2} + C') \tilde{\lambda}^2 N \left(\delta_{\tilde{x}_1, \tilde{x}_2} + \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}}, \end{aligned} \quad (3.16)$$

$$\begin{aligned}
(3.14) : & \left. \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \right\} \leq O(\tilde{\lambda}^2 N) \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \\
(3.15) : & \left. \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \right\} \leq O(\tilde{\lambda}^2 N) \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \\
(3.16) : & \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \leq O(\tilde{\lambda}^2 N) \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \\
(3.17) : & \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \leq O(1) \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \\
(3.18) : & \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \leq O(1) \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \\
(3.19) : & \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array} \leq O(1) \begin{array}{c} \tilde{x}_3 \text{ --- } \tilde{x}_4 \\ \tilde{x}_1 \text{ --- } \tilde{x}_2 \end{array}
\end{aligned}$$

Figure 4: Schematic representations for (3.14)–(3.19). If a line segment between $\tilde{u} = (u, \cdot)$ and $\tilde{v} = (v, \cdot)$ is slashed, then it represents $\tilde{\lambda}/\|u - v\|^{d-2}$; if it is unslashed, then it represents $\delta_{\tilde{u}, \tilde{v}} + \tilde{\lambda}/\|u - v\|^{d-2}$.

or

$$\begin{aligned}
& \sum_{\tilde{v} \in \tilde{\Lambda}_N} \left(\delta_{\tilde{x}_1, \tilde{v}} + \frac{\tilde{\lambda}}{\|x_1 - v\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|v - x_2\|^{d-2}} \left(\delta_{\tilde{x}_3, \tilde{v}} + \frac{\tilde{\lambda}}{\|x_3 - v\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|v - x_4\|^{d-2}} \\
& \leq \left(\delta_{\tilde{x}_1, \tilde{x}_3} + \frac{\tilde{\lambda}}{\|x_1 - x_3\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|x_3 - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}} \\
& \quad + C' \tilde{\lambda}^2 N \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}} \\
& \leq \left(1 + (2^{d-2} + C') \tilde{\lambda}^2 N \right) \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}}. \tag{3.17}
\end{aligned}$$

By similar computations, we can show that there is a $C'' < \infty$ such that

$$\sum_{\tilde{v} \in \tilde{\Lambda}_N} \prod_{j=1}^3 \left(\delta_{\tilde{x}_j, \tilde{v}} + \frac{\tilde{\lambda}}{\|x_j - v\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|v - x_4\|^{d-2}} \leq C'' \left(\delta_{\tilde{x}_1, \tilde{x}_2} + \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \right) \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}}, \tag{3.18}$$

and

$$\sum_{\tilde{v} \in \tilde{\Lambda}_N} \prod_{j=1}^4 \left(\delta_{\tilde{x}_j, \tilde{v}} + \frac{\tilde{\lambda}}{\|x_j - v\|^{d-2}} \right) \leq C'' \left(\delta_{\tilde{x}_1, \tilde{x}_2} + \frac{\tilde{\lambda}}{\|x_1 - x_2\|^{d-2}} \right) \left(\delta_{\tilde{x}_3, \tilde{x}_4} + \frac{\tilde{\lambda}}{\|x_3 - x_4\|^{d-2}} \right). \tag{3.19}$$

Now, we resume the proof of bounding the second and third terms on the right-hand side of (3.11). First, by (2.30) and $\mathcal{J} \epsilon_N^2 N < 1$, we have

$$\sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N} \leq \frac{O(\tilde{\lambda})}{\|x - u\|^{d-2}}. \quad (3.20)$$

Therefore, by (3.15), the second term on the right-hand side of (3.11) is bounded as

$$\begin{aligned} & \langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N} \sum_{\tilde{u}(\neq \tilde{o}), \tilde{v}} \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N}^2 \langle \sigma_{\tilde{x}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N} \\ & \leq \langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N} \sum_{\tilde{u}} \frac{O(\tilde{\lambda})^2}{\|u\|^{2(d-2)}} \left(\delta_{\tilde{x}, \tilde{u}} + \frac{O(\tilde{\lambda})}{\|x - u\|^{d-2}} \right) \frac{O(\tilde{\lambda})}{\|x - u\|^{d-2}} \\ & \leq \left(\delta_{\tilde{o}, \tilde{x}} + \frac{O(\tilde{\lambda})}{\|x\|^{d-2}} \right) \frac{O(\tilde{\lambda}^4 N)}{\|x\|^{2(d-2)}}. \end{aligned} \quad (3.21)$$

Similarly, by (3.15), the third term on the right-hand side of (3.11) is bounded as

$$\begin{aligned} & \sum_{\tilde{u}, \tilde{w}} \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N}^2 \langle \sigma_{\tilde{o}} \sigma_{\tilde{w}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{w}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{w}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N} \\ & \quad \times \sum_{\tilde{v}, \tilde{y}(\neq \tilde{x})} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle \sigma_{\tilde{v}} \sigma_{\tilde{y}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{y}} \sigma_{\tilde{w}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{y}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N}^2 \\ & \leq \sum_{\tilde{u}, \tilde{w}} \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N}^2 \langle \sigma_{\tilde{o}} \sigma_{\tilde{w}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{w}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{w}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N} \\ & \quad \times \sum_{\tilde{y}} \frac{O(\tilde{\lambda})}{\|y - u\|^{d-2}} \left(\delta_{\tilde{w}, \tilde{y}} + \frac{O(\tilde{\lambda})}{\|y - w\|^{d-2}} \right) \frac{O(\tilde{\lambda})^2}{\|x - y\|^{2(d-2)}} \\ & \leq \sum_{\tilde{u}} \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N}^2 \frac{O(\tilde{\lambda}^3 N)}{\|x - u\|^{d-2}} \sum_{\tilde{w}} \langle \sigma_{\tilde{o}} \sigma_{\tilde{w}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{w}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{\tilde{w}} \sigma_{\tilde{x}} \rangle_{\tilde{\Lambda}_N} \frac{O(\tilde{\lambda})}{\|x - w\|^{d-2}}. \end{aligned} \quad (3.22)$$

Then, by applying (3.18) to control the sum over \tilde{w} , and then applying (3.17) to control the sum over \tilde{u} , we obtain (cf., Figure 5)

$$(3.22) \leq \left(\delta_{\tilde{o}, \tilde{x}} + \frac{O(\tilde{\lambda})}{\|x\|^{d-2}} \right) \sum_{\tilde{u}} \langle \sigma_{\tilde{o}} \sigma_{\tilde{u}} \rangle_{\tilde{\Lambda}_N}^2 \frac{O(\tilde{\lambda}^4 N)}{\|x - u\|^{2(d-2)}} \leq \left(\delta_{\tilde{o}, \tilde{x}} + \frac{O(\tilde{\lambda})}{\|x\|^{d-2}} \right) \frac{O(\tilde{\lambda}^4 N)}{\|x\|^{2(d-2)}}. \quad (3.23)$$

Summarizing (3.10)–(3.11) and (3.21)–(3.23) and then using $\tilde{\lambda}^2 N \ll 1$, we conclude that the sum on the right-hand side of (3.10) obeys the inequality (3.7). We finish the proof of Proposition 3.1. \blacksquare

Figure 5: Schematic representations for (3.22)–(3.23). Each inequality gives a bound on the sum over vertices labeled with the letter in the parentheses. The factors $O(\tilde{\lambda}^2 N)$ are omitted for brevity.

3.2 Bound on the two-point function

In this subsection, we prove that the assumed bound on G_N in Proposition 3.1 indeed holds for all $\mu \in (\mu_N, \mu_N^{(0)}]$ (n.b., $\mu_N \leq \mu_N^{(0)}$ is due to Proposition 2.2(i)), where

$$\mu_N = \inf \left\{ \mu : \sum_{x \in \mathbb{Z}^d} G_N(x) < \infty \right\}, \quad (3.24)$$

$$\mu_N^{(0)} = \frac{1}{\epsilon_N^2} \left(\frac{1}{N} - \text{Tanh}^{-1} \frac{1 - N \sum_x \tanh J_{o,x}}{N-1} \right). \quad (3.25)$$

We note that $\mu_N^{(0)}$ is the value of μ at which $p = 1$. Although the exact expression for $\mu_N^{(0)}$ is unimportant, it shows that $\mu_N^{(0)}$ tends to the massless point $\hat{\mathcal{J}}$ for the Gaussian model as $N \rightarrow \infty$. In fact,

$$\mu_N^{(0)} < \frac{1}{\epsilon_N^2} \left(\frac{1}{N} - \frac{1 - N \sum_x \tanh J_{o,x}}{N-1} \right) < \frac{\hat{\mathcal{J}} N}{N-1}. \quad (3.26)$$

For now, we assume $\hat{\mathcal{J}}/2 < \mu_N < \mu_N^{(0)}$, which is to be verified later.

Let

$$\tilde{K} = \sup_{x \in \mathbb{Z}^d} \|x\|^{d-2} S_1(x), \quad (3.27)$$

where we recall that S_1 is the random-walk's Green function generated by the 1-step distribution $D(x) = \tanh J_{o,x} / \sum_y \tanh J_{o,y}$. Notice that, under the assumption in Proposition 3.1 (cf., $\hat{\mathcal{J}} \epsilon_N^2 N < 1$ and $N \geq 2$),

$$J_{o,x} \geq \tanh J_{o,x} \geq J_{o,x} \left(1 - \frac{J_{o,x}^2}{3} \right) \geq J_{o,x} \left(1 - \frac{(\hat{\mathcal{J}} \epsilon_N^2)^2}{3} \right) \geq \frac{11}{12} J_{o,x}, \quad (3.28)$$

and therefore D inherits all the properties of $\mathcal{J}/\hat{\mathcal{J}}$. In particular, there is a $\tau \in (0, 2)$ such that

$$1 - \hat{D}(k) = \frac{|k|^2}{2d} \sum_x |x|^2 D(x) + O(|k|^{2+\tau}), \quad (3.29)$$

$$D^{*n}(x) \leq \frac{O(n)}{\|x\|^{d+2}}, \quad (3.30)$$

$$\left| D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \right| \leq \frac{O(n)|y|^2}{\|x\|^{d+4}} \quad [|y| \leq \tfrac{1}{3}|x|]. \quad (3.31)$$

Applying those bounds to the analysis in [5] for the random-walk's Green function, we can choose the value of \tilde{K} in (3.27) independently of λ, μ, N .

Now, we prove the following theorem.

Theorem 3.3. *Let $d > 4$, $N > \frac{2\hat{\mathcal{J}}^2}{\lambda} \vee \left(\frac{2\mu^2}{\lambda}\right)^3$ and $\lambda \ll 2\mu^2$. Then, for any $\mu \in (\mu_N, \mu_N^{(0)}]$,*

$$\bar{G}_\mu \equiv \sup_x \|x\|^{d-2} G_N(x) \leq 2\tilde{K}. \quad (3.32)$$

Proof. First, we note that $\bar{G}_{\mu_N^{(0)}} \leq \tilde{K}$, due to Proposition 2.2(i). In order to complete the proof, it thus suffices to show

(I) continuity of \bar{G}_μ in $\mu \in (\mu_N, \mu_N^{(0)}]$,

(II) existence of a forbidden region: $\bar{G}_\mu \notin (2\tilde{K}, 3\tilde{K}]$ for every $\mu \in (\mu_N, \mu_N^{(0)})$.

In order to prove (I), we use the following lemma, which is a simple adaptation of [22, Lemma 5.13] to the current setting.

Lemma 3.4. *Let $\{g_\mu(x)\}_{x \in \mathbb{Z}^d}$ be an equicontinuous family of functions in $\mu \in [m, M]$, i.e., for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\mu - \mu'| < \delta \Rightarrow |g_\mu(x) - g_{\mu'}(x)| < \varepsilon$, uniformly in $\mu, \mu' \in [m, M]$ and $x \in \mathbb{Z}^d$. If $\bar{g}_\mu \equiv \sup_x g_\mu(x)$ is finite for each $\mu \in [m, M]$, then \bar{g}_μ is also continuous in $\mu \in [m, M]$.*

In order to apply this lemma to the current setting and prove continuity of \bar{G}_μ , it suffices to show that $\{\|x\|^{d-2} G_N(x)\}_{x \in \mathbb{Z}^d}$ is an equicontinuous family of functions in $\mu \in [m, \mu_N^{(0)}]$, for every $m \in (\mu_N, \mu_N^{(0)})$. However, since $G_N(x)$ is an increasing limit of $G_{\tilde{\Lambda}_N}(o, x)$ as $\Lambda \uparrow \mathbb{Z}^d$, it boils down to show that $\|x\|^{d-2} \frac{d}{d\mu} G_{\tilde{\Lambda}_N}(o, x)$ is bounded uniformly in $x \in \mathbb{Z}^d$, $\Lambda \subset \mathbb{Z}^d$ and $\mu \in [m, \mu_N^{(0)}]$. First, by using Lebowitz' inequality [17],

$$\begin{aligned} \left| \frac{d}{d\mu} G_{\tilde{\Lambda}_N}(o, x) \right| &= \epsilon_N^2 \left| \frac{d}{dI} \left(\frac{1 - (N-1) \tanh I}{N} \langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle_{\tilde{\Lambda}_N} \right) \right| \\ &= \epsilon_N^2 \left| -\frac{N-1}{N \cosh^2 I} \langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle_{\tilde{\Lambda}_N} + \frac{1 - (N-1) \tanh I}{N} \sum_{y \in \Lambda} \sum_{1 \leq i < j \leq N} \langle \tilde{\sigma}_o \tilde{\sigma}_x; \sigma_{(y,i)} \sigma_{(y,j)} \rangle_{\tilde{\Lambda}_N} \right| \\ &\leq \epsilon_N^2 \left(\frac{N-1}{N} \langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle_{\tilde{\Lambda}_N} + \frac{1 - (N-1) \tanh I}{N} \sum_{y \in \Lambda} \sum_{\substack{i, j \in [N] \\ (i \neq j)}} \langle \tilde{\sigma}_o \sigma_{(y,i)} \rangle_{\tilde{\Lambda}_N} \langle \sigma_{(y,j)} \tilde{\sigma}_x \rangle_{\tilde{\Lambda}_N} \right) \\ &\leq \frac{\epsilon_N^2 N}{1 - (N-1) \tanh I} \left(G_{\tilde{\Lambda}_N}(o, x) + \sum_{y \in \Lambda} G_{\tilde{\Lambda}_N}(o, y) G_{\tilde{\Lambda}_N}(y, x) \right) \\ &\leq \frac{1}{\mu} \left(G_N(x) + \sum_{y \in \mathbb{Z}^d} G_N(y) G_N(x - y) \right). \end{aligned} \quad (3.33)$$

By Proposition 2.2(ii), we have $G_N(y - x) \leq \bar{K} / \|y - x\|^{d+\alpha}$, where $\bar{K} = \max_{\mu \in [m, \mu_N^{(0)}]} K_\mu$ and α is an arbitrarily large number in the current setting. Therefore, we arrive at

$$\|x\|^{d-2} \left| \frac{d}{d\mu} G_{\tilde{\Lambda}_N}(o, x) \right| \leq \frac{\bar{K}}{m \|x\|^{2+\alpha}} + \frac{\|x\|^{d-2}}{m} \sum_{y \in \mathbb{Z}^d} \frac{\bar{K}}{\|y\|^{d+\alpha}} \frac{\bar{K}}{\|y - x\|^{d+\alpha}} \leq \frac{C}{\|x\|^{2+\alpha}}, \quad (3.34)$$

where the constant C is independent of Λ and μ . This completes the proof of (I). \blacksquare

Next, we prove (II) by showing that $\bar{G}_\mu \leq 3\tilde{K}$ implies $\bar{G}_\mu \leq 2\tilde{K}$ for each $\mu \in (\mu_N, \mu_N^{(0)})$. First, we derive an identity for G_N using (3.1). Under the assumption $\bar{G}_\mu \leq 3\tilde{K}$ and the hypothesis of the theorem, we can use Proposition 3.1 to obtain the $T \rightarrow \infty$ limit of (3.1):

$$\langle\langle \sigma_{\tilde{o}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N} = \pi_{\tilde{\Lambda}_N}(\tilde{o}, \tilde{x}) + \sum_{\tilde{u}, \tilde{v} \in \tilde{\Lambda}_N} \pi_{\tilde{\Lambda}_N}(\tilde{o}, \tilde{u}) (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{x}} \rangle\rangle_{\tilde{\Lambda}_N}, \quad (3.35)$$

where $\pi_{\tilde{\Lambda}_N}(\tilde{o}, \tilde{x})$ satisfies (3.9). Let

$$\Pi_{\tilde{\Lambda}_N}(o, x) = \sum_{i, j \in [N]} \pi_{\tilde{\Lambda}_N}((o, i), (x, j)). \quad (3.36)$$

which satisfies

$$\left| \frac{\Pi_{\tilde{\Lambda}_N}(o, x)}{N} - \delta_{o, x} \left(1 - \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{o}} \rangle\rangle_{\tilde{\Lambda}_N} \right) \right| \leq O(\tilde{\lambda}^2) \left(\delta_{o, x} + \frac{O(\tilde{\lambda}N)}{\|x\|^{3(d-2)}} \right). \quad (3.37)$$

Then, there is a subsequential limit $\Pi_N \equiv \lim_{\Lambda_j \uparrow \mathbb{Z}^d} \Pi_{\Lambda_j \times [N]}$ such that, for every $x \in \mathbb{Z}^d$,

$$\left| \frac{\Pi_N(x)}{N} - \delta_{o, x} \left(1 - \bigcirc_{\tilde{o}} \right) \right| \leq O(\tilde{\lambda}^2) \left(\delta_{o, x} + \frac{O(\tilde{\lambda}N)}{\|x\|^{3(d-2)}} \right) \stackrel{\tilde{\lambda} < \tilde{\lambda}^2 N}{\leq} \frac{O(\tilde{\lambda}^3 N)}{\|x\|^{3(d-2)}}, \quad (3.38)$$

where

$$\bigcirc_{\tilde{o}} = \sum_{\tilde{v} \in \tilde{\mathbb{Z}}_N^d} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \langle\langle \sigma_{\tilde{v}} \sigma_{\tilde{o}} \rangle\rangle_{\tilde{\mathbb{Z}}_N^d} \stackrel{(2.30)}{=} O(\tilde{\lambda}). \quad (3.39)$$

Therefore, the limit of the sum of (3.35) equals

$$\begin{aligned} \langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\mathbb{Z}}_N^d} &= \Pi_N(x) + \sum_{\tilde{u}, \tilde{v} \in \tilde{\mathbb{Z}}_N^d} \frac{\Pi_N(u)}{N} (\tanh \tilde{J}_{\tilde{u}, \tilde{v}}) \frac{\langle\langle \sigma_{\tilde{v}} \tilde{\sigma}_x \rangle\rangle_{\tilde{\mathbb{Z}}_N^d}}{N} \\ &= \Pi_N(x) + \sum_{u, v \in \mathbb{Z}^d} \Pi_N(u) (\tanh J_{u, v}) \langle\langle \tilde{\sigma}_v \tilde{\sigma}_x \rangle\rangle_{\tilde{\mathbb{Z}}_N^d} \\ &\quad + (N-1)(\tanh I) \sum_{v \in \mathbb{Z}^d} \left(\frac{\Pi_N(v)}{N} - \delta_{o, v} \right) \langle\langle \tilde{\sigma}_v \tilde{\sigma}_x \rangle\rangle_{\tilde{\mathbb{Z}}_N^d} \\ &\quad + (N-1)(\tanh I) \langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\mathbb{Z}}_N^d}. \end{aligned} \quad (3.40)$$

Solving this identity for $\langle\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\rangle_{\tilde{\mathbb{Z}}_N^d}$ and then dividing both sides by N , we obtain

$$\begin{aligned} G_N(x) &= \frac{\Pi_N(x)}{N} + \sum_{u, v \in \mathbb{Z}^d} \frac{\Pi_N(u)}{N} pD(v-u) G_N(x-v) \\ &\quad + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \sum_{v \in \mathbb{Z}^d} \left(\frac{\Pi_N(v)}{N} - \delta_{o, v} \right) G_N(x-v) \\ &= \frac{\Pi_N(x)}{N} + \sum_{v \in \mathbb{Z}^d} F_N(v) G_N(x-v), \end{aligned} \quad (3.41)$$

where

$$F_N(x) = \sum_{u \in \mathbb{Z}^d} \frac{\Pi_N(u)}{N} pD(x-u) + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \left(\frac{\Pi_N(x)}{N} - \delta_{o,x} \right). \quad (3.42)$$

Here, we note that, by summing (3.41) over $x \in \mathbb{Z}^d$,

$$\sum_{x \in \mathbb{Z}^d} G_N(x) = \hat{G}_N(0) = \frac{\hat{\Pi}_N(0)/N}{1 - \hat{F}_N(0)}. \quad (3.43)$$

Since $\hat{\Pi}_N(0)/N > 0$ when $\tilde{\lambda}^2 N \ll 1$ (cf., (3.38)), it must be that $\hat{F}_N(0) < 1$ for $\mu > \mu_N$, which is equivalent to

$$\begin{aligned} \frac{\hat{\Pi}_N(0)}{N} N \sum_{x \in \mathbb{Z}^d} \tanh J_{o,x} + (N-1)(\tanh I) \left(\frac{\hat{\Pi}_N(0)}{N} - 1 \right) &< 1 - (N-1) \tanh I \\ \Leftrightarrow N \sum_{x \in \mathbb{Z}^d} \tanh J_{o,x} + (N-1) \tanh I &< \frac{1}{\hat{\Pi}_N(0)/N}. \end{aligned} \quad (3.44)$$

A sufficient condition for the above inequality is

$$\begin{aligned} N \sum_x J_{o,x} + NI &= 1 + (\hat{\mathcal{J}} - \mu) \epsilon_N^2 N < \frac{1}{\hat{\Pi}_N(0)/N} \\ \Leftrightarrow \mu &> \hat{\mathcal{J}} - \frac{1}{\epsilon_N^2 N} \left(\frac{1}{\hat{\Pi}_N(0)/N} - 1 \right) \\ &= \hat{\mathcal{J}} - \mu \tilde{\lambda} N \left(\bigcirc_{\tilde{o}} + O(\tilde{\lambda}^3 N) \right) = \hat{\mathcal{J}} - \frac{O(\lambda)}{\mu} \\ \Leftrightarrow \mu &> \hat{\mathcal{J}} - \frac{O(\lambda)}{\hat{\mathcal{J}}}. \end{aligned} \quad (3.45)$$

Therefore, the assumed bound $\mu > \hat{\mathcal{J}}/2$ (cf., below (3.26)) is indeed true if $\lambda \ll 1$ (or $\hat{\mathcal{J}} \gg 1$).

Next, we compare (3.41) with the convolution equation for the random-walk's Green function:

$$S_q(x) = \delta_{o,x} + (qD * S_q)(x). \quad (3.46)$$

Inspired by their similarity, we approximate G_N by $r \frac{\Pi_N}{N} * S_q$, with some $r \in (0, \infty)$ and $q \in [0, 1]$. In order to do so, we first rearrange those convolution equations to get

$$\frac{\Pi}{N} = G * (\delta - F), \quad \delta = (\delta - qD) * S_q, \quad (3.47)$$

where, for brevity, we have omitted the subscripts and the spatial variables. Using those identities, we can rewrite G as

$$\begin{aligned}
G &= r \frac{\Pi}{N} * S_q + G * \delta - r \frac{\Pi}{N} * S_q \\
&= r \frac{\Pi}{N} * S_q + G * (\delta - qD) * S_q - rG * (\delta - F) * S_q \\
&= r \frac{\Pi}{N} * S_q + G * E * S_q,
\end{aligned} \tag{3.48}$$

where

$$E = (\delta - qD) - r(\delta - F). \tag{3.49}$$

We choose q and r to satisfy

$$\begin{cases} \hat{E}(0) = 1 - q - r(1 - \hat{F}(0)) = 0, \\ \bar{\nabla}^2 \hat{E}(0) \equiv \lim_{|k| \rightarrow 0} \frac{\hat{E}(0) - \hat{E}(k)}{1 - \hat{D}(k)} = -q + r \bar{\nabla}^2 \hat{F}(0) = 0, \end{cases} \tag{3.50}$$

or equivalently

$$q = r \bar{\nabla}^2 \hat{F}(0), \quad r = \frac{1}{1 - \hat{F}(0) + \bar{\nabla}^2 \hat{F}(0)}. \tag{3.51}$$

Then, we can rewrite E as

$$E = \delta - r(\delta - F + \bar{\nabla}^2 \hat{F}(0)D) = r(-(\hat{F}(0)\delta - F) + \bar{\nabla}^2 \hat{F}(0)(\delta - D)). \tag{3.52}$$

However, since (cf., (3.42))

$$\hat{F}(k) = \frac{\hat{\Pi}(k)}{N} p \hat{D}(k) + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \left(\frac{\hat{\Pi}(k)}{N} - 1 \right), \tag{3.53}$$

we have

$$\hat{F}(0)\delta - F = p \frac{\Pi}{N} * (\delta - D) + \left(p + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \right) \left(\frac{\hat{\Pi}(0)}{N} \delta - \frac{\Pi}{N} \right), \tag{3.54}$$

so that

$$\bar{\nabla}^2 \hat{F}(0) = p \frac{\hat{\Pi}(0)}{N} + \left(p + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \right) \frac{\bar{\nabla}^2 \hat{\Pi}(0)}{N}. \tag{3.55}$$

Therefore,

$$r = \left(1 + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \left(1 - \frac{\hat{\Pi}(0)}{N} + \frac{\bar{\nabla}^2 \hat{\Pi}(0)}{N} \right) + p \frac{\bar{\nabla}^2 \hat{\Pi}(0)}{N} \right)^{-1}, \tag{3.56}$$

and

$$E = r \left(p \left(\frac{\hat{H}(0)}{N} \delta - \frac{H}{N} \right) * (\delta - D) - \left(p + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \right) \left(\frac{\hat{H}(0)}{N} \delta - \frac{H}{N} - \frac{\bar{\nabla}^2 \hat{H}(0)}{N} (\delta - D) \right) \right). \quad (3.57)$$

Due to those rewrites, we can show the following proposition, whose proof follows after completion of the proof of (II).

Proposition 3.5. *Let q and r be chosen to satisfy (3.50)–(3.51). Under the hypothesis of Theorem 3.3, there is a $\rho > 0$ such that*

$$r = 1 - O(\tilde{\lambda}^2 N), \quad 0 \leq 1 - q \leq O(\tilde{\lambda}^2 N), \quad |(E * S_q)(x)| \leq \frac{O(\tilde{\lambda}^2 N)^2}{\|x\|^{d+\rho}}. \quad (3.58)$$

Finally, we can conclude $\bar{G}_\mu \leq 2\tilde{K}$ (hence (II)) by first rewriting (3.48) as

$$G = r \frac{\hat{H}(0)}{N} S_q - r \left(\frac{\hat{H}(0)}{N} \delta - \frac{H}{N} \right) * S_q + G * E * S_q, \quad (3.59)$$

and then applying (3.38), Lemma 3.2 and Proposition 3.5. This completes the proof of Theorem 3.3. \blacksquare

Proof of Proposition 3.5. To evaluate r , we must investigate $(N-1)(\tanh I)/(1 - (N-1)\tanh I)$, p and $\bar{\nabla}^2 \hat{H}(0)/N$ in (3.56). For the first two, it is easy to show that

$$\frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \leq \frac{NI}{1 - NI} \leq \frac{1}{\mu \epsilon^2 N}, \quad (3.60)$$

and that, by using $\hat{\mathcal{J}} < \mu + O(\lambda)/\mu$ (cf., (3.45)),

$$p \leq \frac{N \sum_x J_{o,x}}{1 - NI} = \frac{\hat{\mathcal{J}}}{\mu} = 1 + \frac{O(\lambda)}{\mu^2} = 1 + O(\tilde{\lambda}^2 N). \quad (3.61)$$

On the other hand, since $N > (2\mu^2/\lambda)^3$ (so that $\mu \epsilon^2 N < \tilde{\lambda}^2 N$), we have

$$\begin{aligned} \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} &\geq \frac{(N-1)(I - \frac{1}{3}I^3)}{1 - (N-1)(I - \frac{1}{3}I^3)} \geq \frac{(N-1)I - NI^3}{1 - (N-1)I + NI^3} \\ &\geq \frac{NI - \frac{1}{N} - \frac{1}{N^2}}{1 - NI + \frac{1}{N} + \frac{1}{N^2}} \geq \frac{1 - \mu \epsilon^2 N(1 + 2\tilde{\lambda})}{\mu \epsilon^2 N(1 + 2\tilde{\lambda})} \geq \frac{1 - O(\tilde{\lambda}^2 N)}{\mu \epsilon^2 N}. \end{aligned} \quad (3.62)$$

Moreover, by using (3.26) and $N > 2(\hat{\mathcal{J}} \vee \mu)^2/\lambda$ (so that $\hat{\mathcal{J}}^2 \epsilon^4 < 1/N^2$), we have

$$\begin{aligned} p &\geq \frac{N \sum_x (J_{o,x} - \frac{1}{3}J_{o,x}^3)}{1 - (N-1)(I - \frac{1}{3}I^3)} \geq \frac{\hat{\mathcal{J}} \epsilon^2 N(1 - \hat{\mathcal{J}}^2 \epsilon^4)}{\mu \epsilon^2 N(1 + 2\tilde{\lambda})} \\ &\geq \frac{N-1}{N} (1 - \hat{\mathcal{J}}^2 \epsilon^4) (1 - 2\tilde{\lambda}) \geq 1 - O(\tilde{\lambda}^2 N). \end{aligned} \quad (3.63)$$

Since $1/(\mu\epsilon^2 N) = \tilde{\lambda}N$, we can summarize the above bounds as

$$\frac{(N-1)\tanh I}{1-(N-1)\tanh I} = \tilde{\lambda}N(1 - O(\tilde{\lambda}^2 N)), \quad p = 1 + O(\tilde{\lambda}^2 N). \quad (3.64)$$

For $\bar{\nabla}^2 \hat{H}(0)/N$ in (3.56), we use (3.38) to obtain

$$\left| \frac{\hat{H}(0)}{N} - \frac{\hat{H}(k)}{N} \right| \leq \sum_x (1 - \cos(k \cdot x)) \left| \frac{\Pi(x)}{N} \right| \leq O(\tilde{\lambda}^3 N) \sum_{x \neq o} \frac{1 - \cos(k \cdot x)}{|x|^{3(d-2)}}. \quad (3.65)$$

However, since $d > 4$, there is a $0 < \tau < 2 \wedge (2(d-4))$ such that

$$\sum_{x \neq o} \frac{1 - \cos(k \cdot x)}{|x|^{3(d-2)}} = \frac{|k|^2}{2d} \sum_{x \neq o} \frac{1}{|x|^{d+2(d-4)}} + O(|k|^{2+\tau}). \quad (3.66)$$

Using (3.29), we obtain

$$\frac{\bar{\nabla}^2 \hat{H}(0)}{N} = \lim_{k \rightarrow 0} \frac{\hat{H}(0)/N - \hat{H}(k)/N}{1 - \hat{D}(k)} = \frac{\sum_x |x|^2 \Pi(x)/N}{\sum_x |x|^2 D(x)} = O(\tilde{\lambda}^3 N). \quad (3.67)$$

As a result,

$$r = \left(1 + \tilde{\lambda}N(1 - O(\tilde{\lambda}^2 N)) \left(\bigcirc_{\substack{\bullet \\ \hat{o}}} + O(\tilde{\lambda}^3 N) \right) + O(\tilde{\lambda}^3 N) \right)^{-1} = 1 - O(\tilde{\lambda}^2 N). \quad (3.68)$$

To evaluate $1 - q$ is straightforward. By (3.51) and (3.53), we obtain

$$\begin{aligned} 1 - q &= \underbrace{r(1 - \hat{F}(0))}_{\geq 0 \text{ (cf., (3.43))}} = r \left(1 - \frac{\hat{H}(0)}{N} p - \frac{(N-1)\tanh I}{1-(N-1)\tanh I} \left(\frac{\hat{H}(0)}{N} - 1 \right) \right) \\ &= r \left(1 - p + \left(p + \frac{(N-1)\tanh I}{1-(N-1)\tanh I} \right) \left(\bigcirc_{\substack{\bullet \\ \hat{o}}} + O(\tilde{\lambda}^3 N) \right) \right) = O(\tilde{\lambda}^2 N). \end{aligned} \quad (3.69)$$

Finally, we investigate $(E * S_q)(x)$. First, for a given $T \in (0, \infty)$, we split it into two as

$$\begin{aligned} (E * S_q)(x) &= \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \hat{E}(k) \frac{e^{-ik \cdot x}}{1 - q\hat{D}(k)} \\ &= \int_0^\infty dt \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \hat{E}(k) e^{-t(1-q\hat{D}(k)) - ik \cdot x} \equiv X_{>T} + X_{<T}, \end{aligned} \quad (3.70)$$

where

$$X_{>T} = \int_T^\infty dt \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \hat{E}(k) e^{-t(1-q\hat{D}(k)) - ik \cdot x}, \quad (3.71)$$

$$X_{<T} = \int_0^T dt \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \hat{E}(k) e^{-t(1-q\hat{D}(k)) - ik \cdot x}. \quad (3.72)$$

The value of T is arbitrary for now, but it is to be determined shortly.

Next, we estimate $X_{>T}$ by taking the Fourier transform of (3.57) as

$$\begin{aligned} \hat{E}(k) = r(1 - \hat{D}(k)) & \left(p \left(\frac{\hat{\Pi}(0)}{N} - \frac{\hat{\Pi}(k)}{N} \right) \right. \\ & \left. - \left(p + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \right) \left(\frac{\hat{\Pi}(0)/N - \hat{\Pi}(k)/N}{1 - \hat{D}(k)} - \frac{\bar{\nabla}^2 \hat{\Pi}(0)}{N} \right) \right). \end{aligned} \quad (3.73)$$

By (3.29) and using (3.64)–(3.67) to evaluate the expression in the biggest parentheses of (3.73), we obtain

$$|\hat{E}(k)| \leq O(\tilde{\lambda}^2 N)^2 |k|^{2+\tau}. \quad (3.74)$$

Therefore, by substituting this to (3.71) and using (3.29) and $q \geq 1 - O(\tilde{\lambda}^2 N)$ (cf., (3.69)), we obtain

$$\begin{aligned} |X_{>T}| & \leq \int_T^\infty dt \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} |\hat{E}(k)| e^{-tq(1 - \hat{D}(k))} = O(\tilde{\lambda}^2 N)^2 \int_T^\infty dt t^{-1 - \frac{d+\tau}{2}} \\ & = O(\tilde{\lambda}^2 N)^2 T^{-\frac{d+\tau}{2}}. \end{aligned} \quad (3.75)$$

Let

$$\rho = \frac{2\tau}{d+2+\tau}, \quad T = \|x\|^{2-\rho}. \quad (3.76)$$

Then, we arrive at

$$|X_{>T}| \leq \frac{O(\tilde{\lambda}^2 N)^2}{\|x\|^{d+\rho}}. \quad (3.77)$$

Next, we estimate $X_{<T}$ by first expanding $e^{tq\hat{D}(k)}$ as

$$\begin{aligned} X_{<T} & = \int_0^T dt e^{-t} \sum_{n=0}^\infty \frac{(tq)^n}{n!} \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \hat{E}(k) \hat{D}(k)^n e^{-ik \cdot x} \\ & = \int_0^T dt e^{-t} \sum_{n=0}^\infty \frac{(tq)^n}{n!} (E * D^{*n})(x). \end{aligned} \quad (3.78)$$

Since (3.57) can be rearranged as

$$\begin{aligned} E & = r \left(-p \left(\frac{\hat{\Pi}(0)}{N} \delta - \frac{\Pi}{N} \right) * D - \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \left(\frac{\hat{\Pi}(0)}{N} \delta - \frac{\Pi}{N} \right) \right. \\ & \quad \left. + \left(p + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \right) \frac{\bar{\nabla}^2 \hat{\Pi}(0)}{N} (\delta - D) \right), \end{aligned} \quad (3.79)$$

we have

$$\begin{aligned}
(E * D^{*n})(x) = & r \left(-p \sum_{y \neq o} \frac{\Pi(y)}{N} \left(D^{*(n+1)}(x) - D^{*(n+1)}(x-y) \right) \right. \\
& - \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \sum_{y \neq o} \frac{\Pi(y)}{N} \left(D^{*n}(x) - D^{*n}(x-y) \right) \\
& + \left(p + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \right) \frac{\bar{\nabla}^2 \hat{H}(0)}{N} \\
& \left. \times \sum_{y \neq o} D(y) \left(D^{*n}(x) - D^{*n}(x-y) \right) \right). \tag{3.80}
\end{aligned}$$

Suppose that

$$\left| \sum_{y \neq o} \frac{\Pi(y)}{N} \left(D^{*n}(x) - D^{*n}(x-y) \right) \right| \leq \frac{O(\tilde{\lambda}^3 N)}{\|x\|^{d+2}} \left(\frac{n}{\|x\|^2} + 1 \right), \tag{3.81}$$

$$\left| \sum_{y \neq o} D(y) \left(D^{*n}(x) - D^{*n}(x-y) \right) \right| \leq \frac{O(1)}{\|x\|^{d+2}} \left(\frac{n}{\|x\|^2} + 1 \right), \tag{3.82}$$

so that, by (3.64) and (3.68),

$$|(E * D^{*n})(x)| \leq \frac{O(\tilde{\lambda}^2 N)^2}{\|x\|^{d+2}} \left(\frac{n}{\|x\|^2} + 1 \right). \tag{3.83}$$

Then, by (3.76) and (3.78), we obtain

$$|X_{<T}| \leq \frac{O(\tilde{\lambda}^2 N)^2 T}{\|x\|^{d+2}} \left(\frac{T}{\|x\|^2} + 1 \right) = \frac{O(\tilde{\lambda}^2 N)^2}{\|x\|^{d+\rho}}. \tag{3.84}$$

Combining this with (3.70) and (3.77), we obtain the desired bound on $(E * S_q)(x)$, as in (3.58).

Now, it remains to show (3.81)–(3.82). Since their proofs are almost identical, we only show here (3.81). First, we split the sum into three as

$$\sum_{y \neq o} \frac{\Pi(y)}{N} \left(D^{*n}(x) - D^{*n}(x-y) \right) = \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{3.85}$$

where

$$\Sigma_1 = \sum_{y: 0 < |y| \leq \frac{1}{3}|x|} \frac{\Pi(y)}{N} \left(D^{*n}(x) - D^{*n}(x-y) \right), \tag{3.86}$$

$$\Sigma_2 = \sum_{y: |x-y| \leq \frac{1}{3}|x|} \frac{\Pi(y)}{N} \left(D^{*n}(x) - D^{*n}(x-y) \right), \tag{3.87}$$

$$\Sigma_3 = \sum_{y: |y| \wedge |x-y| > \frac{1}{3}|x|} \frac{\Pi(y)}{N} \left(D^{*n}(x) - D^{*n}(x-y) \right). \tag{3.88}$$

We estimate Σ_2 , Σ_3 and Σ_1 in order, by using (3.30)–(3.31) and (3.38).

For Σ_2 , since $|y| \geq |x| - |x - y| \geq \frac{2}{3}|x|$ and $3(d-2) > d+2$ for $d > 4$, we obtain

$$|\Sigma_2| \leq \frac{O(\tilde{\lambda}^3 N)}{\|x\|^{3(d-2)}} \sum_{y: |x-y| \leq \frac{1}{3}|x|} \left(D^{*n}(x) + D^{*n}(x-y) \right) \leq \frac{O(\tilde{\lambda}^3 N)}{\|x\|^{d+2}} \left(\frac{n}{\|x\|^2} + 1 \right). \quad (3.89)$$

For Σ_3 , we bound both $D^{*n}(x)$ and $D^{*n}(x-y)$ by $O(n)/\|x\|^{d+2}$ and use $3(d-2) > d+2$ again to obtain

$$|\Sigma_3| \leq \frac{O(n)}{\|x\|^{d+2}} \sum_{y: |y| > \frac{1}{3}|x|} \left| \frac{\Pi(y)}{N} \right| \leq \frac{O(\tilde{\lambda}^3 N)}{\|x\|^{d+4}} n. \quad (3.90)$$

For Σ_1 , we first use the \mathbb{Z}^d -symmetry of Π and then use (3.31) to obtain

$$\begin{aligned} |\Sigma_1| &= \left| \sum_{y: 0 < |y| \leq \frac{1}{3}|x|} \frac{\Pi(y)}{N} \left(D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \right) \right| \\ &\leq \frac{O(n)}{\|x\|^{d+4}} \sum_{y \neq o} |y|^2 \left| \frac{\Pi(y)}{N} \right| \leq \frac{O(\tilde{\lambda}^3 N)}{\|x\|^{d+4}} n \sum_{y \neq o} \frac{1}{|y|^{d+2(d-4)}} = \frac{O(\tilde{\lambda}^3 N)}{\|x\|^{d+4}} n. \end{aligned} \quad (3.91)$$

This completes the proof of (3.81), hence the proof of Proposition 3.5. ■

3.3 The linear Schwinger-Dyson equation

Finally, we derive the linear Schwinger-Dyson equation (1.11) and complete the proof of the main theorem.

In the previous subsection, we have proved that, if $d > 4$, λ is sufficiently small and N is sufficiently large, then there is a $c < \infty$, which is independent of λ, μ, N , such that $G_N(x) \leq c/\|x\|^{d-2}$ holds for all $x \in \mathbb{Z}^d$ and $\mu > \mu_N$. Then, by Proposition 3.1, we have (3.38) uniformly in $\mu > \mu_N$. Therefore, by (3.41)–(3.42), we obtain that, for $\mu > \mu_N$,

$$G_N = \frac{\Pi_N}{N} + \left(\frac{\Pi_N}{N} * pD + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \left(\frac{\Pi_N}{N} - \delta \right) \right) * G_N. \quad (3.92)$$

Let

$$\Phi_N(x) = -\epsilon_N^2 N^2 \left(\frac{\Pi_N(x)}{N} - \delta_{o,x} \right) \stackrel{(3.38)}{=} \epsilon_N^2 N^2 \bigcirc_{\delta} \delta_{o,x} + \frac{O(\lambda/\mu^3)}{\|x\|^{3(d-2)}}, \quad (3.93)$$

so that we can rewrite (3.92) as

$$G_N = \delta - \frac{\Phi_N}{\epsilon_N^2 N^2} + \left(pD - \left(pD + \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \delta \right) * \frac{\Phi_N}{\epsilon_N^2 N^2} \right) * G_N. \quad (3.94)$$

Now, we consider the $N \uparrow \infty$ limit of (3.94). First, we claim that $\lim_{N \uparrow \infty} \mu_N = \mu_c$. In order to see this, we recall that

$$G_N(x) = \frac{1 - (N-1) \tanh I}{\epsilon_N^2 N} \epsilon_N^2 \langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle_{\tilde{\mathbb{Z}}^d} \geq 0. \quad (3.95)$$

Since $\epsilon_N^2 \langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle_{\tilde{\Lambda}_N}$ tends as $N \uparrow \infty$ to $\langle \varphi_o \varphi_x \rangle_\Lambda$ that is bounded above by $\langle \varphi_o^2 \rangle_{\mathbb{Z}^d}$ uniformly in x and Λ (due to monotonicity and the Schwarz inequality), we can change the order of the limits to obtain

$$\lim_{N \uparrow \infty} \epsilon_N^2 \langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle_{\tilde{\mathbb{Z}}^d} = \langle \varphi_o \varphi_x \rangle_\mu, \quad (3.96)$$

hence

$$\lim_{N \uparrow \infty} G_N(x) = \mu \langle \varphi_o \varphi_x \rangle_\mu. \quad (3.97)$$

Because of the nonnegativity of G_N , the $N \uparrow \infty$ limit of $\sum_x G_N(x)$ is finite if and only if χ_μ is finite. This implies $\lim_{N \uparrow \infty} \mu_N = \mu_c$.

Suppose that, for every $\mu > \mu_c$, there is a subsequential limit $\Phi_\mu \equiv \lim_{N_j \uparrow \infty} \Phi_{N_j}$ and that it is summable. Since

$$\frac{1}{\epsilon_N^2 N^2} = \sqrt{\frac{\lambda}{2N}}, \quad pD \xrightarrow{N \uparrow \infty} \frac{\mathcal{J}}{\mu}, \quad \frac{(N-1) \tanh I}{1 - (N-1) \tanh I} \underset{N \uparrow \infty}{\sim} \frac{1}{\mu} \sqrt{\frac{\lambda N}{2}}, \quad (3.98)$$

we can take the limit of (3.94) along this subsequence to obtain

$$\mu \langle \varphi_o \varphi_x \rangle_\mu = \delta_{o,x} + \sum_v \left(\mathcal{J}(v) - \frac{\lambda}{2} \Phi_\mu(v) \right) \langle \varphi_v \varphi_x \rangle_\mu, \quad (3.99)$$

which is equivalent to the linear Schwinger-Dyson equation (1.11).

In order to complete the proof, it remains to show existence and summability of the assumed subsequential limit $\Phi_\mu \equiv \lim_{N_j \uparrow \infty} \Phi_{N_j}$. However, since the last term in (3.93) is summable uniformly in N , we only need to show existence of the limit of the first term in (3.93). Notice that, by (2.27) (see (2.28) as well),

$$\begin{aligned} \epsilon_N^2 N^2 \textcircled{\bullet}_o &= \epsilon_N^2 N^2 \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o}, \tilde{v}}) \frac{\frac{1}{N} \langle \tilde{\sigma}_v \tilde{\sigma}_o \rangle_{\tilde{\mathbb{Z}}^d} - \delta_{v,o}}{N - \delta_{v,o}} \\ &= \epsilon_N^2 N (\tanh I) \left(\langle \tilde{\sigma}_o^2 \rangle_{\tilde{\mathbb{Z}}^d} - N \right) + \epsilon_N^2 N \sum_v (\tanh J_{o,v}) \langle \tilde{\sigma}_v \tilde{\sigma}_o \rangle_{\tilde{\mathbb{Z}}^d} \\ &= N (\tanh I) \left(\epsilon_N^2 \langle \tilde{\sigma}_o^2 \rangle_{\tilde{\mathbb{Z}}^d} - \epsilon_N^2 N + \sum_v \frac{\tanh J_{o,v}}{\tanh I} \epsilon_N^2 \langle \tilde{\sigma}_v \tilde{\sigma}_o \rangle_{\tilde{\mathbb{Z}}^d} \right). \end{aligned} \quad (3.100)$$

Since

$$N \tanh I = 1 - \mu \epsilon_N^2 N \xrightarrow{N \uparrow \infty} 1, \quad \epsilon_N^2 N = \sqrt{\frac{2}{\lambda N}} \xrightarrow{N \uparrow \infty} 0, \quad \frac{\tanh J_{o,v}}{\tanh I} \underset{N \uparrow \infty}{\sim} \epsilon_N^2 N \mathcal{J}(v), \quad (3.101)$$

we obtain (cf., (3.96))

$$\lim_{N \uparrow \infty} \epsilon_N^2 N^2 \bigcirc_{\tilde{\phi}} = \langle \varphi_o^2 \rangle_\mu. \quad (3.102)$$

This completes the proof of the main theorem. ■

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References

- [1] M. Aizenman. Geometric analysis of ϕ^4 fields and Ising models. *Comm. Math. Phys.* **86** (1982): 1–48.
- [2] D. Brydges, J. Fröhlich and T. Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.* **83** (1982): 123–150.
- [3] D. Brydges, J. Fröhlich and A. Sokal. The random-walk representation of classical spin systems and correlation inequalities. II. The skeleton inequalities. *Comm. Math. Phys.* **91** (1983): 117–139.
- [4] D. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. *Comm. Math. Phys.* **97** (1985): 125–148.
- [5] L.-C. Chen and A. Sakai. Critical two-point functions for long-range statistical-mechanical models in high dimensions. To appear in *Ann. Probab.*
- [6] C. Cotar and J.-D. Deuschel. Decay of covariances, uniqueness of ergodic component and scaling limit for a class of $\nabla\phi$ systems with non-convex potential. *Ann. Inst. H. Poincaré Probab. Statist.* **48** (2012): 609–908.
- [7] R. Fernández, J. Fröhlich and A.D. Sokal. *Random walks, critical phenomena, and triviality in quantum field theory*. Springer, Berlin (1992).
- [8] J. Fröhlich. On the triviality of $\lambda\varphi_d^4$ theories and the approach to the critical point in $d \geq 4$ dimensions. *Nucl. Phys. B* **200** (1982): 281–296.
- [9] J. Fröhlich, R. Israel, E.H. Lieb and B. Simon. Phase transitions and reflection positivity. I. General theory and long range lattice models. *Comm. Math. Phys.* **62** (1978): 1–34.
- [10] J. Fröhlich, B. Simon and T. Spencer. Infrared bounds, phase transitions and continuous symmetry breaking. *Comm. Math. Phys.* **50** (1976): 79–95.

- [11] K. Gawędzki and A. Kupiainen. Renormalization group for a critical lattice model. *Comm. Math. Phys.* **88** (1983): 77–94.
- [12] T. Hara. Decay of correlations in nearest-neighbour self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.* **36** (2008): 530–593.
- [13] T. Hara, R. van der Hofstad and G. Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.* **31** (2003): 349–408.
- [14] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. *J. Stat. Phys.* **59** (1990): 1469–1510.
- [15] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.* **128** (1990): 333–391.
- [16] M. Heydenreich, R. van der Hofstad and A. Sakai. Mean-field behavior for long- and finite-range Ising model, percolation and self-avoiding walk. *J. Stat. Phys.* **132** (2008): 1001–1049.
- [17] J. Lebowitz. GHS and other inequalities. *Comm. Math. Phys.* **35** (1974): 87–92.
- [18] B.G. Nguyen and W-S. Yang. Triangle condition for oriented percolation in high dimensions. *Ann. Probab.* **21** (1993): 1809–1844.
- [19] A. Sakai. Mean-field critical behavior for the contact process. *J. Stat. Phys.* **104** (2001): 111–143.
- [20] A. Sakai. Lace expansion for the Ising model. *Comm. Math. Phys.* **272** (2007): 283–344.
- [21] B. Simon and R.B. Griffiths. The $(\phi^4)_2$ field theory as a classical Ising model. *Comm. Math. Phys.* **33** (1973): 145–164.
- [22] G. Slade. The lace expansion and its applications. *Lecture Notes in Math.* **1879** (2006).
- [23] A.D. Sokal. An alternate constructive approach to the φ_3^4 quantum field theory, and a possible destructive approach to φ_4^4 . *Ann. Inst. Henri Poincaré Phys. Théorique* **37** (1982): 317–398.